CONDITION THAT A TENSOR BE THE CURL OF A VECTOR *

BY L. P. EISENHART

It is the purpose of this note to establish the following theorem.

THEOREM. *A necessary and sufficient condition that a covariant skew-symmetric tensor A_{ij} in a space of any order n be expressible in terms of n functions φ_i in the form*

$$A_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \tag{1}$$

is that

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0, \quad (i, j, k = 1, \cdots, n). \tag{2}$$

Consider first the case of 3-space. If φ_2 and φ_3 are any two functions such that

$$A_{23} = \frac{\partial \varphi_2}{\partial x^3} - \frac{\partial \varphi_3}{\partial x^2},$$

the conditions of integrability of

$$\frac{\partial \varphi_1}{\partial x^2} = \frac{\partial \varphi_2}{\partial x^1} + A_{12}, \qquad \frac{\partial \varphi_1}{\partial x^3} = \frac{\partial \varphi_3}{\partial x^1} + A_{13}$$

are satisfied in consequence of (2), and the theorem is established for 3-space.

Now we show that, if the theorem is true for n-space, it is true for $(n + 1)$-space. On this assumption equations (1) hold for $i, j = 1, \cdots, n$. For a particular i and j and for $k = n + 1$, equation (2) may be written in the form

$$\frac{\partial}{\partial x^i}\left(A_{jn+1} - \frac{\partial \varphi_j}{\partial x^{n+1}}\right) = \frac{\partial}{\partial x^j}\left(A_{in+1} - \frac{\partial \varphi_i}{\partial x^{n+1}}\right).$$

Hence a function φ_{n+1} is defined by the equations

$$A_{in+1} = \frac{\partial \varphi_i}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^i}, \qquad A_{jn+1} = \frac{\partial \varphi_j}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^j}. \tag{3}$$

* Presented to the Society, September 7, 1922.

Replacing j in (2) by l $(= 1, \cdots, n;\ \neq j)$, we have, by (3),

$$\frac{\partial A_{ln+1}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \varphi_l}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^l}\right).$$

Consequently (1) holds for $i, j = 1, \cdots, n+1$, and the theorem is established. It should be remarked that one of the functions φ_i may be chosen arbitrarily, or what is equivalent, that the functions φ_i are determined to within additive functions $\partial\psi/\partial x^i$, where ψ is an arbitrary function of the x's.

Thus far we have made no use of the fact that A_{ij} are the components of a covariant tensor. If $A'_{\alpha\beta}$ denote the components of the tensor in terms of coordinates x', then

$$A'_{\alpha\beta} = A_{ij}\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}. \tag{4}$$

If Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ denote the Christoffel symbols of the second kind for the respective systems of coordinates x and x' of a Riemannian geometry, then*

$$\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} = \Gamma'^l_{ij}\frac{\partial x^p}{\partial x'^l} - \Gamma^p_{qr}\frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j}.$$

The same equations obtain in the more general case of a geometry of paths, where the functions $\Gamma^i_{\alpha\beta}$ and $\Gamma'^\alpha_{\beta\gamma}$ are the coefficients of the equations of the paths in the two systems of coordinates.† By means of these equations we show that, if the functions A_{ij} satisfy (2), so also do $A'_{\beta\gamma}$ defined by (4). In consequence of the above theorem equation (4) may be replaced by the equation

$$\frac{\partial \varphi_\alpha'}{\partial x'^\beta} - \frac{\partial \varphi_\beta'}{\partial x'^\alpha} = \left(\frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}\right)\frac{\partial x^i}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta} = \frac{\partial \varphi_i}{\partial x'^\beta}\frac{\partial x^i}{\partial x'^\alpha} - \frac{\partial \varphi_j}{\partial x'^\alpha}\frac{\partial x^j}{\partial x'^\beta}$$
$$= \frac{\partial}{\partial x'^\beta}\left(\varphi_i\frac{\partial x^i}{\partial x'^\alpha}\right) - \frac{\partial}{\partial x'^\alpha}\left(\varphi_j\frac{\partial x^j}{\partial x'^\beta}\right).$$

Hence

$$\varphi_\alpha' = \varphi_i\frac{\partial x^i}{\partial x'^\alpha} + \frac{\partial \psi}{\partial x'^\alpha}, \tag{5}$$

* Bianchi, *Lezioni*, vol. 1, p. 64.

† See PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 21.

where ψ is an arbitrary function.

From (5) it is evident that if A_{ij} are defined as the components of the curl of covariant vector, then (2) are necessarily satisfied; but (2) is not a sufficient condition. That this condition is not sufficient was overlooked by me in a recent paper,* and my conclusions in § 5 are not correct. In fact, the skew-symmetric tensor there defined by S_{ij} is given by

$$S_{ij} = \frac{\partial \Gamma^{\alpha}_{\alpha j}}{\partial x^i} - \frac{\partial \Gamma^{\alpha}_{\alpha i}}{\partial x^j},$$

and the functions $\Gamma^{\alpha}_{\alpha i}$ and $\Gamma'^{\alpha}_{\alpha i}$ in two sets of coordinates are in the relation

$$\Gamma'^{\alpha}_{\alpha i} = \Gamma^{\alpha}_{\alpha j}\frac{\partial x^j}{\partial x'^i} + \frac{\partial}{\partial x'^i}\log \Delta,$$

where Δ is the Jacobian of the transformation.

PRINCETON UNIVERSITY

A NEW GENERALIZATION OF TCHEBYCHEFF'S STATISTICAL INEQUALITY

BY B. H. CAMP

1. *Introduction.* If $f(x)$ is any frequency distribution, and s its standard deviation, the symbol $P(\lambda s)$ may be used to represent the probability that a datum drawn from this distribution will differ from the mean value by as much as λs, numerically. For the solution of various statistical problems it is desirable to have a formula which will measure $P(\lambda s)$ when $f(x)$ is only partially known. A case of practical importance occurs when $f(x)$ represents the distribution of values of a statistical constant determined by sampling from a known distribution, such a constant as, for example, a mean value, or a coefficient of correlation. In such cases it is usually difficult or impossible to find the complete distribution $f(x)$, but quite feasible to find its lower moments. Tchebycheff's well known inequality is: $P(\lambda s) \leqq 1/\lambda^2$. It has been general-

* PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 236.

ized in a formula first discovered by K. Pearson:*

$$P(\lambda s) \leqq \frac{\beta_{2r-2}}{\lambda^{2r}}, \qquad (r = 1, 2, \cdots),$$

where $\beta_{2r-2} = m_{2r}/s^{2r}$, m_{2r} being the $2r$th moment of $f(x)$ about its mean. Pearson's view is that, although in most cases this is a closer inequality than Tchebycheff's, it is usually not close enough to be of practical assistance. However, it is the best formula so far obtained which logically can be used with distributions whose larger even β's depart considerably from their Gaussian values. It is proposed to exhibit here a method by which the right side of Pearson's inequality may usually be decreased by about fifty per cent. The theory will be explained in § 2, and illustrated in § 3.

2. *Theory.* LEMMA. *Let $Q(t) \geqq 0$, and be monotonic, decreasing, and let $d^2Q/dt^2 \geqq 0$, in the interval $1 \leqq t \leqq k$. Let the straight line $y = a + bt$ pass through the points of Q for which $t = 1$ and $t = p$. Then*

$$\mathrm{I} \equiv \int_1 t^{2r-1}Q(t)dt \geqq \int_1 t^{2r-1}ydt \equiv \mathrm{II},$$

if $r \geqq 1$, $k \geqq 2$, $p = 2rk/(2r + 1)$.

It will be observed that, by subtracting a constant from each side of the inequality, the proposition can be reduced to the case where $Q(k) = 0$. This being supposed done, let $h = Q(1)/Q(p)$, and note that by definition $p > 1$, $h \geqq 1$. After determination of the values of a and b in accordance with the conditions imposed on y in the hypothesis, II becomes

$$\mathrm{II} = Q(p)\int_1^k t^{2r-1}\frac{hp - 1 + (1 - h)t}{p - 1}\,dt. \tag{1}$$

Consider now the function represented by the line tangent to the curve Q at $t = p$, that is,

$$z = Q(p) + Q'(p)(t - p). \tag{2}$$

Since Q is concave upwards it never crosses this tangent, and so $Q(t) \geqq z(t)$, and

$$\mathrm{I} \equiv \int_1^k t^{2r-1}Q(t)dt \geqq \int_1^k t^{2r-1}z(t)dt \equiv \mathrm{III}. \tag{3}$$

* BIOMETRIKA, vol. 12 (1919), p. 284.

In order to prove that I $\geqq$ II it is by (3) sufficient to show that III $\geqq$ II, and the lemma will now be established in this manner. First, fix arbitrarily all the parameters that occur in III, viz.: p, r, k, $Q(p)$, and $Q'(p)$, thus fixing the value of III. The value of II is also thus fixed, except so far as variation due to h is concerned. But, from (1),

$$\frac{d(\text{II})}{dh} = \frac{Q(p)(1-k)}{(p-1)(2r+1)} \leqq 0,$$

and therefore, so far as variation with h is concerned, II is greatest when h is least, that is, when $h = 1$. But, then, $Q(1) = Q(p)$, and, since $Q(t)$ is monotonic, $Q(t) = Q(p)$ throughout the interval $1 \leqq t \leqq p$, and so $Q'(p) = 0$. Substituting this value for $Q'(p)$ in (2) and (3), and $h = 1$ in (1), we learn finally that, for this special value of h which makes II greatest, II = III. So, in general, II $\leqq$ III.

THEOREM: (*a*) *Let $f(x)$ be any frequency distribution, and suppose the origin and units to be so chosen that zero is its mean value and*

$$P(x) = \int_x^\infty f(x)dx.$$

(*b*) *Let $f(x)$ be a monotonic, decreasing function of $|x|$ when $|x| \geqq cs$, $c \geqq 0$.* This is a real restriction on $f(x)$, which varies in its severity according to the value of c. It will be discussed later. Its general effect is to except distributions of more than one mode, notably the "U" distributions.

(*c*) *Let $f(x)$ be symmetrical with respect to the centroid ordinate.* This may be assumed without loss of generality, for, if $g(x)$ is a skew distribution whose mean is at $x = 0$, it is clear that $f(x) = \frac{1}{2}[g(x) + g(-x)]$ has the same mean, even moments, and the same values of $P(\lambda s)$ as $g(x)$. *Then*

$$P(\lambda s) \leqq \frac{\beta_{2r-2}}{\left(\lambda \frac{2r+1}{2r}\right)^{2r}} \cdot \frac{1}{1+\phi} + \theta, \tag{4}$$

where

$$(5) \qquad \theta = \frac{P(cs)}{1+\frac{1}{\phi}}, \qquad \phi = \frac{\left(\frac{c}{\lambda}\cdot\frac{2r}{2r+1}\right)^{2r}}{(2r+1)\left(\frac{\lambda}{c}-1\right)},$$

quantities which can be tabulated and are usually very small.

Since $c \geqq 0$, we may write

$$\frac{\beta_{2r-2}}{c^{2r}} \geqq \int_{cs}^{\infty}\left(\frac{x}{cs}\right)^{2r} f(x)dx = -\left[\left(\frac{x}{cs}\right)^{2r}\int_x^{\infty} f(t)dt\right]_{cs}^{\infty} + \int_{cs}^{\infty}\frac{2rx^{2r-1}}{(cs)^{2r}}dx\int_x^{\infty} f(t)dt$$

$$\geqq P(cs) + \frac{2r}{cs}\int_{cs}^{kcs}\left(\frac{x}{cs}\right)^{2r-1}P(x)dx = P(cs) + 2r\int_1^k t^{2r-1}Q(t)dt,$$

where $x = tcs$, $Q(t) = P(tcs)$. Therefore, by the lemma, and by use of the relation $hQ(p) = Q(1)$,

$$\frac{\beta_{2r-2}}{c^{2r}} \geqq P(cs) + \frac{2rQ(p)}{p-1}\int_1^k t^{2r-1}[(hp-1)+(1-h)t]dt$$

$$= \frac{P(cs)}{1+2r-2rk} + P(pcs)\left[\frac{(2r+1)(1-k^{2r})+2r(k^{2r+1}-1)}{1+2r-2rk}\right].$$

After the substitutions $k = p(2r+1)/2r$, $p = \lambda/c$, and some reductions, the result (4) follows from the above inequality. The assumption (*b*) will now be discussed.

Case I: ($c = 0$). It was stated under assumption (*c*) that $f(x)$ might be assumed symmetrical, because a symmetrical function $f(x)$ having the same essential characteristics could be constructed from any skew function $g(x)$ that might arise in practice. It should be noticed, however, that if $g(x)$ is skew, the resulting $f(x)$ may be saddle shaped, with a geometrical minimum at its mean and two symmetrically situated maxima on either side. In such a case, the value of c would equal the distance from the mean to either maximum, s being the unit of measurement. In other words, if skewness be defined as (mean − mode)/s, c *may* be as large as the absolute value of the skewness. Now, skew functions are usually the ones that occur in practice, and so one could not usually

assert, a priori, that $c = 0$ for a given distribution. This embarrassment might be avoided by choosing the origin at the mode of the skew function instead of at the mean, and by defining the moments and $P(\lambda s)$ with reference to such an origin, in which case, it could be shown, the above theorem would apply, and, for unimodal functions, c would equal zero.* But this is often impracticable, and certainly not customary. So it will be necessary to consider larger values of c. The following table for the case $c = 0$ is of value, however, since it is involved in the computation for the cases where $c > 0$. When $c = 0$, $\phi = \theta = 0$, and formula (4) is exactly Pearson's, except for the factor $[(2r + 1)/2r]^{2r}$ in the denominator. The values of this factor are as follows:

Formula	r	Factor
β_0	1	2.25
β_2	2	2.44
β_4	3	2.52

Formula	r	Factor
β_6	4	2.56
β_8	5	2.59
β_{10}	6	2.60

CASE II: ($c = 1$). If functions of more than one mode be excepted, a skewness greater than unity is very rare. Therefore, this may be regarded as a strong case, and Case III below, where $c = 2$, as an extreme case. To compute formula (4) we may use the table under Case I for the first fraction, and the tables below for ϕ and θ. The values given for θ are really the extreme values which θ has when $P(cs = s) = 1$. It is unnecessary to estimate them more closely. In this case, the theorem will be found of value only when $\lambda > 1$, but this is not a serious misfortune, since in practice it is most needed for about the range $2 < \lambda < 4$.

r	θ			ϕ		
	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$
1	.015 6	.008 13	.004 28	.015 8	.008 23	.004 84
2	.001 40	.000 51	.000 22			
3	.000 15	.000 039	.000 012	ϕ equals θ to the number of places indicated		
4	$1.9 \cdot 10^{-5}$	$3.8 \cdot 10^{-6}$	$7.7 \cdot 10^{-7}$			

*While this paper was in press a paper by B. Meidel appeared, COMPTES RENDUS, vol. 175, p. 806 (Nov. 6, 1922), giving the result of this paper for the case $c = 0$, and using the mode as origin.

CASE III: ($c = 2$). The computation in this case will again be performed by use of the table under Case I, and of the subjoined table for ϕ and for $\theta/P(cs = 2s)$. Then Pearson's inequality should be used to estimate $P(2s)$. In this case the theorem will be found of value only when $\lambda > 2$.

r	$\theta/P(2s)$			ϕ		
	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$
1	.275	.116	.060 6	.379	.132	.065
2	.118	.031 4	.011 5	.134	.032	.012
3	.056 1	.009 88	.002 63	.059	.010	.003
4	.028 3	.003 38	.006 58	.029	.003	.001

3. *Example.* In BIOMETRIKA * the frequency distributions of a number of coefficients of correlation are actually computed. One may apply the above inequalities to these distributions (using the β's as if they only, not the entire frequency distributions, had been calculated), and then compare the results obtained with the true values of $P(\lambda s)$ to be found by partially summing the frequencies there tabulated. Take $\lambda = 3$ in the distribution for which $\rho = 0.8$, $n = 100$, on page 403. If the distribution be regarded as a set of rectangles, the true value of $P(3s)$ is about 0.009. The β_2 formula shows that $P(3s) \leqq 0.018$, and the β_4 formula that $P(3s) \leqq 0.012$. Case II has been used on the theory that the practical statistician would be confident that $c \leqq 1$, but the use of Case III would increase the better of the two results given to only 0.014. Here, although β_2 departs from its Gaussian value by only 0.42, it is not safe to use the Gaussian table, which would make $P(3s) = 0.0027$. The reason is that the higher β's depart much more widely from their Gaussian values. They are, roughly, $\beta_2 = 3.42$, $\beta_4 = 22.7$, $\beta_6 = 290$, $\beta_8 = 5000$, $\beta_{10} = 100\,000$; instead of 3, 15, 105, 945, and 10 395, respectively. It is a common, but unwarranted and sometimes very faulty practice, to use the Gaussian where β_2 only is near its Gaussian value.

WESLEYAN UNIVERSITY

* Vol. 11 (1915–17), pp. 379–404.

TWO THEOREMS ON MULTIPLE INTEGRALS*

BY PHILIP FRANKLIN

1. *Introduction.* The multiple integrals in question here were discussed by Poincaré (ACTA MATHEMATICA, vol. 9, p. 321) in a paper in which he defined them and derived their integrability conditions. In this note we give two theorems, which simplify the derivation of these conditions, and enable us to express the integral of an integrable function over an open region as an integral of one order lower over the boundary of this region. The second theorem was suggested by a special case proved by Professor Eisenhart.

2. *First Theorem.* Our first theorem is

$$\int n \int \Sigma A_{i_1 \cdots i_n} dx_{i_1} \cdots dx_{i_n} = \frac{1}{m-n} \int n+1 \int \Sigma \frac{\partial A_{i_1 \cdots i_n}}{\partial x_{i_{n+1}}} dx_{i_{n+1}} dx_{i_1} \cdots dx_{i_n}, \tag{1}$$

where the n (dimensional) region of integration for the left member is the boundary of the $n+1$ region for the right member, and the summations apply to all n or $n+1$ permutations of the m subscripts of our fundamental space. The theorem is proved by adding equations of the form

$$\int n \int \tilde{A}_{1 \cdots n} dx_1 \cdots dx_n = \int n+1 \int \frac{\partial A_{1 \cdots n}}{\partial x_{n+1}} dx_{n+1} dx_1 \cdots dx_n, \tag{2}$$

which may be established when the A's are functions of m variables by methods similar to those ordinarily used to prove (2) in n-space. The equation (1) shows that the set of equations

$$\sum_j (-1)^{j-1} \frac{\partial A_{i_1 \cdots i_{j-1} i_{j+1} \cdots i_{n+1}}}{\partial x_{i_j}} = 0 \tag{3}$$

is a necessary and sufficient condition for the vanishing of the left member of (1) for all closed regions of integration, i.e.,

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that (3) is the condition of integrability for the form

$$\sum A_{i_1 \ldots i_n} dx_{i_1} \cdots dx_{i_n}. \tag{4}$$

3. *Second Theorem.* It follows from (1) also that the integral

$$\int n+1 \int \sum B_{i_1 \ldots i_{n+1}} dx_{i_1} \cdots dx_{i_{n+1}}, \tag{5}$$

taken over any open region may be expressed as an integral

$$(m-n)(n+1) \int n \int \sum A_{i_1 \ldots i_n} dx_{i_1} \cdots dx_{i_n}, \tag{6}$$

taken over its boundary, provided A's can be found such that

$$B_{i_1 \ldots i_{n+1}} = \sum_j (-1)^{j-1} \frac{\partial A_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_{n+1}}}{\partial x_j}. \tag{7}$$

We shall show that *a necessary and sufficient condition for the existence of such a set of A's is that* (5) *be integrable,* i.e., that

$$\sum (-1)^{j-1} \frac{\partial B_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_{n+2}}}{\partial x_j} = 0. \tag{8}$$

This theorem is known for the case $n = 0$. We shall outline an induction by which the general case may be reduced to this one. We first consider the case where there are $n+2$ i's, and hence $n+2$ equations (7) and one condition (8). If we write

$$B_{1 \ldots p-1,\ p+1 \ldots n+2} = B'_{1 \ldots p-1,\ p+1 \ldots n+1} + (-1)^n \frac{\partial A_{1 \ldots p-1,\ p+1 \ldots n+1}}{\partial x_{n+2}}, \tag{9}$$

where $p \neq n+2$, and where the B''s are defined by the corresponding equation (7), and if we determine the A's appearing explicitly in (9) so as to satisfy the remaining equation of the set (7),

$$B_{1 \ldots n+1} = \sum_j (-1)^{j-1} \frac{\partial A_{1 \ldots j-1,\ j+1 \ldots n+1}}{\partial x_j}, \tag{10}$$

but leave their values otherwise arbitrary, we find that by the use of (9) and (10) the condition (8) reduces to the form

$$\sum_j (-1)^{j-1} \frac{\partial B'_{1 \ldots j-1,\ j+1 \ldots n+1}}{\partial x_j} = 0. \tag{11}$$

Since (11) is of the same form as (8), but in one less index,

and since the expressions for the B's in terms of A's are similar to (7), we have reduced the problem to one of lower order.

To complete our proof, we reduce the case for $m+1$ subscripts to that for m. We first obtain values for the A's not involving one subscript, say k, from the equations and conditions not involving k, by the case assumed as the basis of the induction. Then we make the substitutions:

$$B_{i_1 \ldots i_n k} = B'_{i_1 \ldots i_n} + (-1)^n \frac{\partial A_{i_1 \ldots i_n}}{\partial x_k}, \tag{12}$$

which effects the desired reduction.

This proves the sufficiency of the conditions; that they are necessary follows by direct substitution. A more complete discussion of the properties of multiple integrals is given in an expository article that will appear in the ANNALS OF MATHEMATICS.

HARVARD UNIVERSITY

KIRKMAN PARADES*

BY F. N. COLE

On examining the complete list of non-equivalent triad systems in 15 letters published in the MEMOIRS OF THE NATIONAL ACADEMY OF SCIENCES (vol. 14, No. 2, pp. 77–80), it appears that only four of these are Kirkman systems, if this name be applied to those cases where the 35 triads divide into seven sets (or columns) of five with each column containing all the 15 letters. Such a seven-column arrangement might be called a Kirkman parade. And it turns out that three of the Kirkman systems give each two non-equivalent parades, while the fourth system gives only one parade.

Kirkman proposed his problem in the LADY'S AND GENTLEMAN'S DIARY for 1850. The seven solutions were correctly given by Woolhouse in the same Diary for 1862 and 1863. In 1881 Carpmael published a list of eleven solutions in the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY (vol. 12, pp. 148–156). But his sixth and seventh items duplicate the third and fourth, and the fifth and eleventh duplicate the ninth and tenth.

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In forming the columns of a parade, any two of them can always be reduced to one of the types

	1	2	3		1	4	7			1	2	3		1	4	7
	4	5	6		2	5	8			4	5	6		2	5	10
(α)	7	8	9		3	10	13		(β)	7	8	9		3	8	13
	10	11	12		6	11	14			10	11	12		6	11	14
	13	14	15		9	12	15			13	14	15		9	12	15

We may say that two columns are *laced* in the mode (α) or the mode (β), and the seven parades are readily distinguished by this lacing: The parades follow in tabular form, together with the substitution group of each system and parade and the scheme of interlacing of the columns.

I–II

1	2	3	1	4	7	1	5	15	1	9	13	1	6	10	1	8	11	1	12	14
4	5	6	2	5	8	2	9	10	2	4	12	2	11	15	2	7	14	2	6	13
7	8	9	3	10	13	3	4	14	3	5	11	3	7	12	3	6	9	3	8	15
10	11	12	6	11	14	6	8	12	6	7	15	4	8	13	4	10	15	4	9	11
13	14	15	9	12	15	7	11	13	8	10	14	5	9	14	5	12	13	5	7	10

												1	6	10	1	8	11	1	12	14
												2	7	14	2	6	13	2	11	15
"			"			"			"			3	8	15	3	7	12	3	6	9
												4	9	11	4	10	15	4	8	13
												5	12	13	5	9	14	5	7	10

These two parades are included in a single triad system, which contains 120 conjugates of each of them. The group of the system is of order 8!/2. The group of I is of order 168 and is generated by (1 11 7 5 9 12 2)(3 8 13 10 14 15 4) and (1 13 12 11 15 3 4)(2 8 14 7 9 5 10); it is transitive in all the letters but 6. The group of II is also of order 168, but it is transitive in seven and in eight letters, being generated by (1 2 9 13 6 3 10)(4 14 8 11 15 12 5) and (1 10 13 9 2 6 3)(4 8 7 14 11 5 15).

In both I and II the columns have only the lacing (α).

III–IV

1	2	3	1	4	7	1	5	15	1	9	10	1	6	13	1	8	11	1	12	14
4	5	6	2	5	8	2	9	13	2	4	12	2	7	14	2	6	10	2	11	15
7	8	9	3	10	13	3	4	11	3	5	14	3	8	12	3	7	15	3	6	9
10	11	12	6	11	14	6	7	12	6	8	15	4	10	15	4	9	14	4	8	13
13	14	15	9	12	15	8	10	14	7	11	13	5	9	11	5	12	13	5	7	10

												1	6	13	1	12	14	1	8	11
												2	11	15	2	6	10	2	7	14
"			"			"			"			3	8	12	3	7	15	3	6	9
												4	9	14	4	8	13	4	10	15
												5	7	10	5	9	11	5	12	13

These are included in a single triad system, which contains 12 conjugates of each of them. The group of the system is of order 288. The group of III is of order 24 and is generated by (1 8 5)(2 15 11)(3 6 9)(4 10 12)(7 14 13) and (1 10 2 13)(5 15 4 12)(7 11 8 14)(6 9). The group of IV is also of order 24 and is generated by (1 10 13)(3 6 9)(4 5 11)(8 15 12) and (1 12 2 15)(4 11 5 14)(7 10 8 13)(3 9). The last column in each case is in relation (α) to all the other columns; each of the latter has two (α) lacings and four (β) lacings.

V–VI

1	2	3	1	4	7	1	5	13	1	6	8	1	9	10	1	11	15	1	12	14
4	5	6	2	5	10	2	4	12	2	11	13	2	7	14	2	6	9	2	8	15
7	8	9	3	8	13	3	9	11	3	7	12	3	5	15	3	4	14	3	6	10
10	11	12	6	11	14	6	7	15	4	10	15	4	8	11	5	8	12	4	9	13
13	14	15	9	12	15	8	10	14	5	9	14	6	12	13	7	10	13	5	7	11
												1	9	10	1	11	15	1	12	14
												2	8	15	2	7	14	2	6	9
	"			"			"			"		3	4	14	3	6	10	3	5	15
												5	7	11	4	9	13	4	8	11
												6	12	13	5	8	12	7	10	13

These again are both contained in one triad system. They have both the same group as the system, viz., the tetrahedral group generated by (1 12)(2 7)(5 15)(8 11)(6 10)(9 13) and (1 2 3)(4 12 7)(6 10 9)(5 11 8). The last three columns of each have the lacing (α); the other lacings are all of type (β).

VII

1	2	3	1	4	7	1	5	9	1	6	15	1	8	11	1	10	13	1	12	14
4	5	6	2	5	10	2	4	14	2	9	11	2	6	13	2	7	12	2	8	15
7	8	9	3	8	13	3	10	15	3	7	14	3	4	12	3	6	9	3	5	11
10	11	12	6	11	14	6	8	12	4	8	10	5	7	15	4	11	15	6	7	10
13	14	15	9	12	15	7	11	13	5	12	13	9	10	14	5	8	14	4	9	13

This has a group of order 21, generated by (1 2 12 3 7 4 14)(5 8 11 9 15 10 6) and (2 7 12)(3 4 14)(5 6 11)(8 9 15), which is also the group of the triad system. The lacings of the columns are all of type (β).

COLUMBIA UNIVERSITY

IMPOSSIBILITY OF RESTORING UNIQUE FACTORIZATION IN A HYPER-COMPLEX ARITHMETIC

BY L. E. DICKSON

1. *Introduction.* Most numbers $a + be$, where a and b are integers and $e^2 = 0$, admit of several factorizations into indecomposable numbers. It is proved in § 3 that we cannot restore unique factorization by defining hypercomplex ideals analogous to algebraic ideals, nor (§ 4) by the introduction of any sort of ideals obeying the laws of arithmetic. L. G. du Pasquier* has made statements, omitting proofs, concerning the failure of unique factorization after introducing ideals, apparently meaning those analogous to algebraic ideals.

2. *Hypercomplex Integers.* Consider the hypercomplex numbers $x = a + be$ with rational coordinates a, b, where $e^2 = 0$. Thus $(x - a)^2 = 0$. This quadratic equation has integral coefficients if and only if a is integral. As our integral hypercomplex numbers we shall take those of an infinite system of numbers $a + be$, where a is integral and b rational, such that the system has a basis† 1, ce, i.e., is composed of their linear combinations with integral coefficients. Since we may take ce as a new unit e whose square is zero, we may assume that $a + be$ is integral if and only if a and b are both integers.

* VIERTELJAHRSSCHRIFT, ZÜRICH, vol. 54 (1909), pp. 116–148.
L'ENSEIGNEMENT, vol. 17 (1915), pp. 340–3; vol. 18 (1916), pp. 201–260.
NOUVELLES ANNALES, (4), vol. 18 (1918), pp. 448–461.
COMPTES RENDUS DU CONGRÈS INTERNATIONAL (Strasbourg), 1921.

† We obtain uninteresting results if we omit the assumption of a basis and call $a + be$ integral if a is integral and b rational. It is a unit if $a = \pm 1$. If $r \neq 0$, $\rho = r + se$ is "associated" with its product r by the unit $1 - es/r$. Hence the classes of associated numbers whose real coordinates are not zero are in (1, 1) correspondence with the real integers and obey the laws of divisibility of integers. But se is associated only with $\pm se$. Now te is divisible by every $r + se$, $r \neq 0$, the quotient being et/r.

The units (integral divisors of 1) are $\pm 1 + be$, where b is integral. A simple example shows that the laws of divisibility fail. Let p be any odd prime. Then

$$(1) \qquad p \cdot p = (p + ke)(p - ke),$$

and $p + te$ is indecomposable and is associated with $p + le$ if and only if $t \equiv l \pmod{p}$. The product of $p - ke$ by the unit $1 + e$ is $p + (p - k)e$. Hence if we give to k the values $0, 1, \cdots, \frac{1}{2}(p-1)$ in (1), we obtain the $\frac{1}{2}(p+1)$ essentially different ways of factoring p^2 into indecomposables. As a generalization of (1),

$$(p + ke)(p + le) = (p + xe)(p + ye), \qquad x + y = k + l.$$

3. *Hypercomplex Ideals.* As in the theory of algebraic numbers, define an ideal to be an infinite set of our hypercomplex integers which is closed under addition and subtraction and is such that the product of any number of the set by any hypercomplex integer is equal to a number of the set. Since $(r + se)e = re$, every ideal contains a number te, where $t \neq 0$. Let m be the minimum positive integer such that me is in the ideal. The products xme of me by all hypercomplex integers $x + ye$ constitute a principal ideal, denoted by $[me]$. Consider an ideal I which contains xme, where x ranges over all integers, and further numbers $r_i + s_ie$, where each $r_i \neq 0$. The positive greatest common divisor r of the r_i is a linear combination of them with integral coefficients. The same linear combination of the $r_i + s_ie$ is a number $r + se$ of I. Write q_i for the integer r_i/r. Then I contains

$$r_i + s_ie - q_i(r + se) = k_ie, \qquad k_i \equiv s_i - q_is.$$

Hence $k_i = p_im$, where p_i is an integer. Thus

$$r_i + s_ie = p_i(me) + q_i(r + se),$$

so that I is composed of linear combinations of me and $r + se$ with integral coefficients (i.e., has a basis). Finally, the product of every number of I by every hypercomplex integer must belong to I, which will be true if the products by e belong to I, i.e., if re is in I. Thus $r = qm$, where q is an integer. Hence

$$(2) \quad I = [me, qm + se], \qquad q > 0,\ m > 0,\ 0 \leqq s < m,$$

where the bracket signifies all the linear combinations of the two enclosed numbers with integral coefficients.

The product of two ideals is defined to be the totality of linear combinations with integral coefficients of the various products of numbers of the first ideal by numbers of the second. Hence

$$[te]I = [tqme] = [te]P, \qquad P \equiv [qme,\ qm].$$

The ideal P is distinct from I unless $q = 1$, $s = 0$. Hence the laws of arithmetic do not hold for our ideals.*

The preceding special difficulty may be obviated by excluding one-based ideals $[te]$. Hence we shall supplement our definition of an ideal by making the assumption that it contains numbers which are not divisors of zero, i.e., multiples of e. Now every ideal is of the form (2).

If $q = 1$, (2) is composed of the products of $m + se$ by all hypercomplex integers and hence is called a principal ideal $\{m + se\}$. If $q > 1$, (2) is not a principal ideal. We shall call q the *mass* of the ideal (2). Hence an ideal is a principal ideal if and only if its mass is unity.

The mass of a product of two ideals is the least common multiple of their masses. For, the product of (2) by

$$J = [ne,\ pn + te]$$

is

$$IJ = [p \cdot mne,\ q \cdot mne,\ pqmn + ke], \qquad k = qmt + pns.$$

Let G be the greatest common divisor of $p = PG$ and $q = QG$, whence P and Q are relatively prime. Then G is a linear combination of p and q. The same linear combination of the first two entries in IJ is $Gmne$. Hence

$$IJ = [Gmne,\ PQGGmn + ke],$$

whose mass is PQG, i.e., the least common multiple of p and q.

It follows at once that the mass of the product of any number of ideals is the least common multiple of their masses. The latter is unity if and only if all the masses are unity. Hence a product is a principal ideal if and only if each factor is a principal ideal. In other words, every factor of a principal ideal is itself a principal ideal.

* More simply since $[e][e]$ is not an ideal.

Consequently our ideals fail to explain our difficulty (1), or the equivalent in principal ideals:

$$(1') \qquad \{p\}\cdot\{p\} = \{p + ke\}\{p - ke\}.$$

Since we saw that $p + te$ is an indecomposable number, we conclude that $\{p + te\}$ is not a product of principal ideals and not a product of any ideals. The relation (1′) between distinct indecomposable ideals shows that our ideals do not obey the laws of arithmetic and do not explain our difficulty (1).

4. *Impossibility of the Restoration of Unique Factorization.* In § 3 we saw the futility of the introduction of hypercomplex ideals defined essentially as in the theory of algebraic numbers. We shall now prove that it is impossible to restore unique factorization by the introduction of ideals of any kind such that a number and its products by the units all correspond to a unique ideal (provided the number be not a divisor of zero) and such that the product of two numbers corresponds to the product of the corresponding ideals.

The numbers $a = 3$, $b = 3 + e$, $c = 3 + 2e$ are indecomposable and no two are associated. We have

$$(3) \qquad ac = b^2, \qquad bc = a^2(1 + e), \qquad ab = c^2(1 - e).$$

Let α, β, γ be the distinct ideals $\neq 1$ which correspond uniquely to a, b, c, respectively. Since $1 + e$ and $1 - e$ are units, we have

$$(4) \qquad \alpha\gamma = \beta^2, \qquad \beta\gamma = \alpha^2, \qquad \alpha\beta = \gamma^2.$$

A prime ideal divisor $\delta \neq 1$ of α must divide β and γ. Write

$$\alpha = \alpha_1\delta, \ \beta = \beta_1\delta, \ \gamma = \gamma_1\delta.$$

Then

$$(5) \qquad \alpha_1\gamma_1 = \beta_1^2, \qquad \beta_1\gamma_1 = \alpha_1^2, \qquad \alpha_1\beta_1 = \gamma_1^2.$$

No one of $\alpha_1, \beta_1, \gamma_1$ is unity. For, if $\alpha_1 = 1$, then $\beta_1\gamma_1 = 1$, whence $\beta_1 = \gamma_1 = 1$, whereas α, β, γ are distinct. Since the relations (5) are entirely similar to relations (4), it is impossible to restore, in a finite number of steps, unique factorization in (3) by the introduction of ideals.

5. *Factorization.* There is only a finite number of ways of factoring into indecomposables and each way involves only a finite number of factors. First, to obtain all pairs of factors of te, express t as a product of two integers x, w in all possible ways; the factors are $x + ye, we$, where $0 \leqq y < x$, if we retain only one of associated numbers. To factor $a + be$, where $a > 0$, express a in all possible ways as a product xz of two integers > 1. For each such pair x, z, and for $0 \leqq y < x$, $a + be$ has the factors $x + ye$ and $z + we$, where w is determined uniquely by $xw + yz = b$, the case of a fractional w being excluded.

The only pairs of factors of p^k ($k \geqq 2$), no one a unit, are

$$p^r + ye, \qquad p^{k-2r}(p^r - ye),$$

where r ranges over the positive integers $\leqq \frac{1}{2}k$, and $0 \leqq y < p^r$. The only pairs of factors of $p^k + se$ are

$$p^r + ye, \qquad p^{k-r} + (\sigma - p^{k-2r}y)e,$$

where $r \leqq \frac{1}{2}k$ and p^r divides $s = \sigma p^r$.

If p and q are distinct primes and $k \geqq l$, the only pairs of factors not units of $p^k q^l$ are J, $p^{k-2r}q^{l-2s}K$, where

$$J = p^r q^s + ye, \qquad K = p^r q^s - ye,$$

$0 \leqq r \leqq \frac{1}{2}k$, $0 \leqq s \leqq \frac{1}{2}l$, $0 \leqq y < p^r q^s$; and $p^{k-2r}J$, $q^{l-2s}K$ ($r < \frac{1}{2}k$). In particular pq has only the factors p, q.

Regarding factors of J, note that $pq + se$ has the unique factors, apart from unit factors, $p + ye$ and $q + we$, where $0 \leqq y < p$, $qy \equiv s \pmod{p}$, and w is determined by $pw + yq = s$. Next, $p^2q + se$ has the factors $p^2 + ye$ and $q + we$, where w is uniquely determined by $0 \leqq w < q$, $p^2w \equiv s \pmod{q}$, and y is then determined by $p^2w + qy = s$. It has no further pairs of factors if s is prime to p. But if $s = Sp$, the only additional pairs are the p pairs $p + ye$, $pq + we$, where $0 \leqq y < p$, $w = S - qy$.

THE UNIVERSITY OF CHICAGO

A REVISION OF THE BERNOULLIAN AND EULERIAN FUNCTIONS*

BY E. T. BELL

1. *Introduction.* To secure simplicity and uniformity in the derivation of relations between the Bernoullian and Eulerian *functions* (not to be confused with the *numbers* of the same name) occurring most frequently in applications we take as fundamental a set of four functions instead of the usual pair, and from these by easy substitutions obtain the values of the functions defined by other writers.†

Following Lucas ‡ we use the even suffix notation for the numbers B, G, E, R of Bernoulli, Genocchi, Euler and Lucas; with the exceptions $B_1 = -\frac{1}{2}$, $G_1 = 1$, the numbers of odd rank vanish, and $B_0 = 1$, $G_0 = 0$, $E_0 = 1$, $R_0 = \frac{1}{2}$. Unless otherwise stated n is an arbitrary integer $\geqq 0$. The "representative" or umbral calculus of Blissard § is used throughout, so that the nth power a^n of the umbra a represents the ordinary a_n. The letters x, y, z, u, v denote ordinary algebraic quantities, or ordinaries; $a, b, c, B, E, G, R, \varphi, \psi, \lambda, \mu$ are umbræ. The umbral sin ax, cos ax are defined by the series, assumed absolutely convergent for some $|x| > 0$,

$$\sin ax = \sum_0^\infty (-1)^n a^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_0^\infty (-1)^n a_{2n+1} \frac{x^{2n+1}}{(2n+1)!};$$

$$\cos ax = \sum_0^\infty (-1)^n a^{2n} \frac{x^{2n}}{(2n)!} = \sum_0^\infty (-1)^n a_{2n} \frac{x^{2n}}{(2n)!};$$

the umbral multinomial theorem gives the expansion of $(ax + by + \cdots + cz)^n$ in either of the identical forms

* Presented to the Society, April 8, 1922.

† Accounts of the history and notations of these functions are contained in papers by Worpitzky, CRELLE'S JOURNAL, vol. 94 (1883), pp. 203–232, and Glaisher, QUARTERLY JOURNAL, vol. 29 (1898), pp. 1–169; vol. 42 (1911), pp. 86–157.

‡ *Théorie des Nombres*, Chap. XIV.

§ QUARTERLY JOURNAL, vols. 6–9 (1863–1867); cf. also Lucas, loc. cit., Chap. XIII.

$$\sum \frac{n!}{\alpha!\beta!\cdots\gamma!} x^\alpha y^\beta \cdots z^\gamma a^\alpha b^\beta \cdots c^\gamma,$$

$$\sum \frac{n!}{\alpha!\beta!\cdots\gamma!} x^\alpha y^\beta \cdots z^\gamma a_\alpha b_\beta \cdots c_\gamma,$$

the $\sum$ extending to all sets of values of $\alpha, \beta, \cdots, \gamma \geqq 0$ whose sum is n. When precisely r of the umbræ $a, b, \cdots, c$ are the same they are replaced by r distinct umbræ until after the degradation of exponents. Thus $(a - a)^2$ is obtained from $(a - b)^2$ or $a^2b^0 - 2a^1b^1 + a^0b^2$, and its value is $2(a_0a_2 - a_1^2)$. Throughout the paper $f(x)$ is an arbitrary analytic function of x; its derivatives with respect to x are denoted by accents, $f'(x), f''(x), \cdots$; with respect to any other variable, say u, of which it is a function, by $f_u, f_{uu}, \cdots$. As a special case of importance in applications the power series representation $k_0 + k_1x + k_2x^2 + \cdots$ of $f(x)$ may terminate; that is, for m a finite constant, $k_j = 0, j > m$.

2. *Umbral Identities.* The functions defined by

$$\varphi_n(a, b) \equiv (a + b)^n + (a - b)^n,$$

$$\psi_n(a, b) \equiv (a + b)^n - (a - b)^n,$$

and the identities connecting them, are fundamental for the whole subject.* Each is said to be of rank n. Noting that they may be written umbrally $\varphi^n(a, b), \psi^n(a, b)$, we have

$$\begin{aligned} 2\cos ax \cos bx &= \cos \varphi(a, b)x, \\ 2\sin ax \sin bx &= -\cos \psi(a, b)x; \end{aligned} \tag{1}$$

$$\begin{aligned} 2\sin ax \cos bx &= \sin \varphi(a, b)\, x, \\ 2\cos ax \sin bx &= \sin \psi(a, b)x. \end{aligned} \tag{2}$$

From the definitions we write down as special cases the values of $\varphi_n(xa, yb), \psi_n(xa, yb), \varphi_n(x, yb)$, etc. For, if $a \equiv yb$, then

$$a_n = a^n = y^n b^n = y^n b_n,$$

since y is an ordinary, b an umbra; while if $a \equiv x$,

$$a_n = a^n = x^n.$$

The suffix u denoting as above the u-derivative, we have

* This is true whatever definition of the Bernoullian and Eulerian functions be taken. Thus the existing treatments could be much simplified by the use of φ, ψ. It appears best, however, to recast the entire theory.

$$\lambda_u{}^n(u, a) = n\lambda^{n-1}(u, a), \qquad \lambda = \varphi, \psi; \tag{3}$$

and it is easily seen that

$$\varphi^{n+1}(u, a) = (n+1)\int_0^u \varphi^n(u, a)du + [1-(-1)^n]a_{n+1}. \tag{4}$$

Let ϵ denote either $+1$ or -1. Then

$$(x + \epsilon a + \epsilon b)^n \pm (x + \epsilon a - \epsilon b)^n = (x + \epsilon\lambda(a, b))^n = \sum_{r=0}^{n} \binom{n}{r} \epsilon^r \lambda^r(a, b) x^{n-r},$$

where $\lambda = \varphi$ or ψ according as the upper or lower sign on the left is taken. Multiply this throughout by $k_0, k_1, k_2, \cdots$ and add all the results:

$$f(x + \epsilon a + \epsilon b) \pm f(x + \epsilon a - \epsilon b) = f(x + \epsilon\lambda(a, b)), \tag{5}$$

the ambiguous sign being determined as above. In the same way,

$$f(x + a + b) + \epsilon_1 f(x + a - b) + \epsilon_2 f(x - a - b) + \epsilon_1\epsilon_2 f(x - a + b) = f(\mu(x, \lambda)), \tag{6}$$

where $\lambda = \varphi(a, b)$ or $\psi(a, b)$ according as $\epsilon_1 = +1$ or -1, and $\mu = \varphi$ or ψ according as $\epsilon_2 = +1$ or -1. Many more relations of a similar kind can be written down in the like manner. The process by which (5), (6) are obtained is more easily applied in many cases than the final formulas. For example, suppose we have formed the relation $(a + b)^{2n} = c^{2n}$. Then since n is an arbitrary integer $\geqq 0$ we have identically

$$(x + a + b)^n + (x - a - b)^n = (x + c)^n + (x - c)^n,$$

and from this, proceeding as before,

$$f(x + a + b) + f(x - a - b) = f(x + c) + f(x - c).$$

All of the above formulas can be easily generalized to the case of any number n of umbræ by extending (1), (2) to n factors on the left. In doing this it is advantageous to take as fundamental the function generated by the product of n_1 umbral sines and n_2 umbral cosines, $n_1 + n_2 = n$. The cases $(n_1, n_2) = (1, 1), (2, 0), (0, 2)$ are sufficient for the present sketch.

From Lucas,* we transcribe the relations

$$(7) \qquad x \operatorname{ctn} x = \cos 2Bx, \qquad 2x \tan x = \cos 2Gx,$$

$$(8) \qquad \sec x = \cos Ex, \qquad x \csc x = 2 \cos Rx.$$

Although the lack of symmetry in these destroys many uniformities later we shall retain the definitions of B_n, G_n, E_n, R_n implied in them, as Lucas' notation is well established.

3. *The Fundamental Functions.* The Bernoullian functions $\beta^n(u) \equiv \beta_n(u)$, $\gamma^n(u) \equiv \gamma_n(u)$ of rank n and argument u are now defined by

$$(9) \qquad \beta^n(u) = \varphi^n(u, 2B) = 2 \sum_{r=0} 2^{2r} \binom{n}{2r} B_{2r}\, u^{n-2r},$$

$$(10) \qquad \gamma^n(u) = \varphi^n(u, 2G);$$

and the Eulerian functions similarly by

$$(11) \qquad \eta^n(u) = \varphi^n(u, E) = 2 \sum_{r=0} \binom{n}{2r} E_{2r}\, u^{n-2r},$$

$$(12) \qquad \rho^n(u) = \varphi^n(u, R).$$

When there can be no confusion the argument u will be omitted but understood, and we shall write $\lambda_n(u) \equiv \lambda_n$ $(\lambda = \beta, \gamma, \eta, \rho)$. The derivatives are obtained from (3),

$$(13) \qquad \lambda_u{}^n = n\lambda^{n-1}, \qquad (\lambda = \beta, \gamma, \eta, \rho);$$

and from (9)–(12), noting the zero values of B, G, E, R to write down those of β, γ, η, ρ, we have for $(\lambda, L) \equiv (\beta, B)$, (γ, G) and $(\mu, M) \equiv (\eta, E)$, (ρ, R),

$$(14) \qquad \begin{aligned} \lambda_{2n+1} &= (2n+1) \int \lambda_{2n} du, \\ \lambda_{2n+2} &= (2n+2) \int \lambda_{2n+1} du + 2^{2n+3} L_{2n+2}; \end{aligned}$$

$$(15) \qquad \begin{aligned} \mu_{2n+1} &= (2n+1) \int \mu_{2n} du, \\ \mu_{2n+2} &= (2n+2) \int \mu_{2n+1} du + 2M_{2n+2}. \end{aligned}$$

With the initial values

$$(16) \qquad \beta_0 = 2, \qquad \gamma_0 = 0, \qquad \eta_0 = 2, \qquad \rho_0 = 1,$$

* Loc. cit., p. 261.

these enable us to calculate the successive λ_n rapidly when the B, G, E, R numbers are given by either tables or recurrences. From (9)–(12), (7)–(8), (1), (2) we write down the generating identities,*

(17) $\sin \beta x = 2x \operatorname{ctn} x \sin ux, \qquad \cos \beta x = 2x \operatorname{ctn} x \cos ux;$

(18) $\sin \gamma x = 4x \tan x \sin ux, \qquad \cos \gamma x = 4x \tan x \cos ux;$

(19) $\sin \eta x = 2 \sec x \sin ux, \qquad \cos \eta x = 2 \sec x \cos ux;$

(20) $\sin \rho x = x \csc x \sin ux, \qquad \cos \rho x = x \csc x \cos ux.$

In conjunction with (1), (2), (7), (8) these contain the entire theory of the relations between $\beta, \gamma, \eta, \rho$. The relations are found uniformly from (17)–(20) by obvious trigonometric manipulations. There is space here only for a few examples to illustrate the processes.

4. *Relations between the Functions.* Multiply the identities (17) throughout by $4x \tan x$, use the second of (7) to reduce the left-hand members of the results to the form of the corresponding member of (2), and apply (2):

$$\sin \varphi(\beta, 2G)x = 8x^2 \sin ux, \qquad \cos \varphi(\beta, 2G)x = 8x^2 \cos ux.$$

Equate coefficients of like powers of x,

$$\varphi^n(\beta, 2G) = -8 \frac{d^2u^n}{du^2};$$

generalize this as in obtaining (5), noticing that $f''(x+u) = f_{uu}(x+u)$:

(21) $f(x+\beta+2G) + f(x+\beta-2G) = -8f''(x+u).$

Similarly from (18),

(22) $f(x+\gamma+2B) + f(x+\gamma-2B) = -8f''(x+u).$

If in this $f(x) \equiv x^n$, and then $x = 0$, the form from which it was inferred is recovered. In the same way from (19), (20) after multiplication throughout by $2 \cos x$, $2 \sin x$ respectively,

(23) $f(x+\eta+1) + f(x+\eta-1) = 4f(x+u),$

(24) $f(x+\rho+1) - f(x+\rho-1) = 2f'(x+u);$

and from (17), (18) for the multipliers $\sin x$, $\cos x$ we find first

$$\psi^n(\beta, 1) = 2\varphi_u{}^n(u, 1), \qquad \varphi^n(\gamma, 1) = -4\psi_u{}^n(u, 1),$$

* Note the unnecessary lack of symmetry caused by the unfortunate definitions (7), (8).

whence by generalization to functions f,

$$(25)\quad f(x+\beta+1)-f(x+\beta-1)=2f'(x+u+1)+2f'(x+u-1),$$

$$(26)\quad f(x+\gamma+1)+f(x+\gamma-1)=4f'(x+u-1)-4f'(x+u+1).$$

Again, dividing each of the identities in each of (17)-(20) by the other we get, for $\lambda = \beta, \gamma, \eta, \rho$,

$$\cos \lambda x \tan ux = \sin \lambda x, \qquad \sin \lambda x \cot ux = \cos \lambda x,$$

and hence as before,

$$(27)\quad \varphi^{2n}(\lambda, 2Gu) = -4u\lambda_u{}^{2n}, \qquad \varphi^{2n+1}(\lambda, 2Bu) = 2u\lambda_u{}^{2n+1}.$$

These express the functions of given rank linearly in terms of functions of the same kind and lower ranks. To generalize we proceed as in § 2, getting

$$(28)\quad \begin{aligned} &f(x+\lambda+2Gu)+f(x-\lambda-2Gu)\\ &\quad +f(x+\lambda-2Gu)+f(x-\lambda+2Gu)\\ &\qquad = -4u[f_u(x+\lambda)+f_u(x-\lambda)], \end{aligned}$$

$$(29)\quad \begin{aligned} &f(x+\lambda+2Bu)-f(x-\lambda-2Bu)\\ &\quad +f(x+\lambda-2Bu)-f(x-\lambda+2Bu)\\ &\qquad = 2u[f_u(x+\lambda)-f_u(x-\lambda)], \end{aligned}$$

which hold for $\lambda = \beta, \gamma, \eta, \rho$. The identities from which these are inferred may be recovered as indicated above.

A relation between functions f whose arguments contain at most r distinct umbræ chosen from among $\beta, \gamma, \eta, \rho$ is called r-fo!d. It would be of interest to write out a set of such relations for $r = 1, 2, 3, 4$ containing 4, 6, 4, 1 relations of the respective types (the case $r = 1$ is treated above); but to save space we indicate the method for $r > 1$ in one case only. When $r = 2$ the simplest relations are found by multiplying together two of the generating identities and proceeding as above. Thus (19), (20) give

$$\begin{aligned} \cos \psi\,(\eta, \rho)x &= -4\cos 2Rx + 2\cos 2\rho x,\\ \cos \varphi\,(\eta, \rho)x &= \quad 4\cos 2Rx + 2\cos 2\rho x; \end{aligned}$$

and hence

$$(\eta \pm \rho)^{2n} = 2^{2n+1}\rho^{2n}, \quad 2^{2n+2}R^{2n},$$

the upper sign giving the first on the right, the lower the second. Generalizing we get

(30) $f(x+\eta+\rho)+f(x-\eta-\rho)=2f(x+2\rho)+2f(x-2\rho)$,

(31) $f(x+\eta-\rho)+f(x-\eta+\rho)=4f(x+2R)+4f(x-2R)$.

Other simple types arise from division; thus

$$f(x+\eta+2B)+f(x+\eta-2B)=4f(x+\rho).$$

5. *Functions of Argument* $2u$, $2u-1$. In many applications the values of the functions in § 3 are required only for the arguments $2u$, $2u-1$. It is then desirable to consider the functions of $2u$, $2u-1$ as new functions of u. Accordingly we write the definitions

$$\lambda_j^n(u) \equiv \lambda_{j,n}(u) = \lambda^n(2u-j), \qquad (j=0, 1),$$

for $\lambda = \beta, \gamma, \eta, \rho$. There can be no confusion between the constant suffix j and the u in $\lambda_{ju}{}^n$ denoting the u-derivative of $\lambda_j{}^n$ (the argument u being omitted as before). We have $\lambda_j{}^n(u) = (\lambda(2u)-j)^n$, as is evident on expanding by Taylor's theorem and noting (3). The generating identities are written down from (17)–(20) by changing u into $2u-j$ and replacing λ by λ_j. Hence all the relations between the $\lambda_j{}^n$ ($j=0, 1$) functions of either kind can be written down immediately from those for the λ^n functions by replacing in the latter relations λ by λ_j and u by $2u$ or $2u-1$ according as $j=0$ or 1. It is therefore unnecessary to repeat any of the derivations.

We note one relation of particular interest, the analogue for the $\lambda_1{}^n(u)$ ($\lambda = \beta, \gamma, \eta, \rho$) of relations found by Raabe, Schlömilch and Glaisher for their functions, the obviousness of which by the symbolic method is striking:

(32) $$\lambda_1{}^n(u) = (-1)^n\lambda_1{}^n(1-u), \quad (\lambda = \beta, \gamma, \eta, \rho).$$

Another type may be noticed. Expand $\sin(2u-1)x$, $\cos(2u-1)x$ in the generating relations by the addition theorems for the sine and cosine, use (17), (18) to reduce the results, and finally equate coefficients of like powers of x. Thus

$$\begin{aligned}\sin\beta_1 x &= 2x \operatorname{ctn} x\,[\sin 2ux\cos x - \cos 2ux \sin x]\\ &= 2x \csc x \sin 2ux - 2x\cos(2u-1)x,\\ &= 2\sin\rho(2u)x - 2x\cos(2u-1)x,\end{aligned}$$

and similarly for $\cos\beta_1 x$ and the rest. We find hence

$$\beta_1^n(u) = 2\rho^n(2u) - 2n(2u-1)^{n-1}, \tag{33}$$

$$\gamma_1^n(u) = -2n\eta^{n-1}(2u) + 4n(2u-1)^{n-1}, \tag{34}$$

$$2n\eta_1^{n-1}(u) = \gamma^n(2u) + 4n(2u)^{n-1}, \tag{35}$$

$$2\rho_1^n(u) = \beta^n(2u) - 2n(2u)^{n-1}; \tag{36}$$

and generalizing,

$$f(x+\beta_1) = 2f(x+\rho_0) - f_u(x+2u-1), \tag{37}$$

$$f(x+\gamma_1) = -f_u(x+\eta_0) + 2f_u(x+2u-1), \tag{38}$$

$$f_u(x+\eta_1) = f(x+\gamma_0) + 2f_u(x+2u), \tag{39}$$

$$2f(x+\rho_1) = f(x+\beta_0) - f_u(x+2u). \tag{40}$$

Thus if a relation between any members of either set λ_j ($\lambda = \beta, \gamma, \eta, \rho$, $j = 0, 1$) be given, a relation between the corresponding members of the other set can be written down from it, β, ρ and γ, η being the pairs of correspondents.

6. *Values of the* λ_j^n ($j = 0, 1$) *for Integral Arguments* > 0. The connection between our functions and those of other writers is obtained (among other ways) by noting the sums of like powers of integers to which the λ_j^n degenerate when the argument u is an integer > 0. There are 16 cases in sets of four each, obtained by multiplying $\sin 2ux$, $\cos 2ux$, $\sin (2u-1)x$, $\cos (2u-1)x$ by x ctn x, $2x$ tan x, sec x, x csc x, reducing the products to linear functions of circular functions (of which not more than one in any case is other than a sine or cosine), and taking the coefficient of $x^{2n}/(2n)!$ or $x^{2n+1}/(2n+1)!$ according as the expanded product is an even or an odd function of x. These coefficients are simply expressible in terms of λ_j ($j = 0, 1$) functions; they also are readily expressed in terms of the Bernoullian and Eulerian functions already in the literature. A full discussion of them having been given elsewhere it suffices here to state the reference.*

University of Washington

* Messenger of Mathematics, vol. 50 (1921), pp. 177–186. The functions φ, ψ of the paper differ slightly from those of the present, also β, γ there have no connection with β, γ here.

ON THE EXISTENCE OF CURVES WITH ASSIGNED SINGULARITIES*

BY J. L. COOLIDGE

The fundamental relations which connect the point and line singularities of algebraic plane curves are called Plücker's equations; they give necessary limitations upon the numbers of characteristics of the different sorts. These equations do not, however, contain in themselves any existence theorems, and after a solution in integers has been obtained, there is no guarantee that there is any curve whose characteristics are the numbers found. Erroneous views have been held on this point. The fact that a quartic is possible with three cusps suggests the existence of a unicursal curve of any order all of whose singular points are cusps; such a curve can not exist.

Various attempts have been made to show the existence of curves with assigned singularities, the most important being that of Lefschetz.† As far as simple nodes are concerned, he completed the solution, exhibiting the existence of plane curves with no singularities but simple nodes, and these in any desired number up to the theoretical maximum. He also showed that the requirement of each additional node imposed just one new condition. With regard to cusps, he was less successful. He established certain upper limits which may be attained, but the conclusions are not clean-cut, and depend upon what he calls the *Postulate of singularities*, which consists essentially in assuming that when we require a certain curve to have an additional cusp, we do not thereby impose upon it automatically more than one additional cusp.

The present paper follows closely Lefschetz's methods, but reaches a more definite and much simpler conclusion, and does so without the use of his postulate, which is proved in the course of the work. The final result is as follows.

* Presented to the Society, December 27, 1922.

† *On the existence of loci with given singularities*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), p. 23.

THEOREM. *The maximum possible number of cusps for a curve of odd order n and deficiency p, where* $n \geqq 2p + \sqrt{8p+9}$, *is*

$$\frac{3}{2}[(n-2)+2p]-\frac{1}{2}\cdot$$

The maximum number for a curve of even order n and deficiency p, where $n \geqq 2p+1+\sqrt{8p+1}$, *is*

$$\frac{3}{2}[(n-2)+2p]\cdot$$

There exist curves of order n and deficiency p having cusps in any number up to the maximum.

Let us begin by writing some of Plücker's numbers, and the equations which connect them:

Order of curve	$= n$,	Number of cusps	$= \kappa$,
Class of curve	$= m$,	Deficiency	$= p$,
Number of nodes	$= \delta$,	Number of inflections	$= i$.

We have, then, the following equations of Plücker:

(1) $$m = n(n-1) - 2(\delta+\kappa) - \kappa,$$

(2) $$p = \frac{(n-1)(n-2)}{2} - (\delta+\kappa),$$

(3) $$\begin{aligned} m &= 2(n-1)+2p-\kappa, \\ n &= 2(m-1)+2p-i, \\ i &= 3n(n-2)-6(\delta+\kappa)-2\kappa, \\ i &= 3[(n-2)+2p]-2\kappa, \end{aligned}$$

(4) $$\kappa = \frac{3}{2}[(n-2)+2p]-\frac{i}{2}\cdot$$

In order to prove that a curve may have the maximum number of cusps we must show that we can have a curve of given order and deficiency with one inflection, if the order be odd, and no inflection if it be even.

Suppose that we have a unicursal curve of order n with a multiple point A of order $n-1$, all the tangents being distinct. Let B be a simple point of this curve. Pass a quadric surface through A and B, and let V be the point where a generator through A meets a generator through B. Let us project our

plane curve upon this quadric from V. The space curve will not pass through V, but will meet VA at the points of contact of the tangent planes through the tangents to the plane curve at A, and will meet VB in the point of contact of the tangent plane through the tangent at B to the plane curve. Since the space curve does not go through V, and meets a plane through V in n points other than V, it is a space curve of order n. Now if we project this space curve back upon the plane from a point which does not lie on one of its tangents, or on the common secant of two intersecting tangents, we get a plane curve, also of order n, whose only singularities are nodes, and this plane curve is unicursal, for it is birationally related to the first one. But if the point of projection lie upon a tangent to the space curve the new plane curve will have just one cusp, but a deficiency 0 as before. We see, therefore, that we may have a unicursal curve with no singularities but nodes,* or with a single cusp. The classes in these two cases are, by (3),

$$2(n-1); \qquad 2(n-1)-1.$$

If we increase n to $n+1$, we shall get the classes

$$2n; \qquad 2n-1.$$

Thus it appears that we can find a unicursal curve with one cusp and any odd class above 1, or with no cusp and any even class. Dually, we can find a curve which is unicursal with one inflection and any odd order, or with no inflection, and any even order, and this shows that we can find a unicursal curve with the maximum number of cusps. Now for a curve of deficiency p, we follow a general form of reasoning of Lefschetz.

There will exist a curve of deficiency p and the maximum number of cusps if there exist a curve of given order and deficiency with one inflection or no inflection, as the case may be, i.e., if there exist a curve of given deficiency and class with one cusp or no cusp, or, finally, a curve of given order

* See Snyder, *Construction of curves of given deficiency*, this BULLETIN, vol. 15 (1908), p. 1.

and deficiency with one cusp or no cusp. Since the number of double tangents cannot be negative, when m and n are even, $i = 0$, and

$$(m-1)(m-2) > 2p, \qquad 2(m-1) \geqq 1 + \sqrt{8p+1},$$

$$n \geqq 2p + 1 + \sqrt{8p+1};$$

when n is odd, $i = 1$, and

$$n \geqq 2p + \sqrt{8p+9}.$$

What is the contrary hypothesis? It is that if a curve have deficiency p and given order it must have more than one cusp or the equivalent in higher singularities. If this were true for every curve of deficiency p it would be true for every curve of lesser deficiency, for the curves of lesser deficiency have more singularities, i.e., they have imposed upon them all the conditions for the curves of higher deficiency, and others also. But this last hypothesis does not hold where the deficiency is 0; hence it does not hold in the general case. Thus, there are curves of any deficiency with one cusp or no cusps, or with one inflection or no inflections, hence curves of any deficiency with the maximum number of cusps.

We have now, lastly, to prove that we can have any number of cusps less than the maximum. We can have no cusps at all or one cusp. Start with this and impose conditions for one cusp, then another, etc. We shall thus get any number of cusps up to the maximum unless it happen that at some stage of the process the imposition of an additional cuspidal condition will produce several new cusps. If this should happen, the intervening number of cusps would be impossible. But another result would be reached also, for the number of additional conditions imposed in order to change from a curve with no cusps or one cusp to one with the maximum number of cusps would be less than the number of cusps so gained.

If a curve have the order n and κ cusps, where this number is one or zero, according as n is odd or even, the number of degrees of freedom is

$$\frac{n(n+3)}{2} - \left[\frac{(n-1)(n-2)}{2} - p\right] - \kappa = 3n + p - 1 - \kappa.$$

Hence the number of degrees of freedom of a curve of class m with one inflection or no inflections is

$$3m + p - 1 - i,$$

where $i = 0$ or $i = 1$.

If the conditions for cusps are all independent, and only in that case, the amount of freedom would be

$$\frac{n(n+3)}{2} - \left[\frac{(n-1)(n-2)}{2} - p\right] - \frac{3}{2}[(n-2) + 2p] + \frac{i}{2} = \frac{3n}{2} + 2 - 2p + \frac{i}{2}.$$

But

$$m = 2(n-1) + 2p - \frac{3}{2}[(n-2) + 2p] + \frac{i}{2} = \frac{n}{2} + 1 - p + \frac{i}{2}.$$

Hence

$$\frac{3n}{2} + 2 - 2p + \frac{i}{2} = 3m + p - 1 - i.$$

It appears, then, that a curve with the maximum number of cusps has exactly the freedom compatible with the appearance of one new cusp for each additional cuspidal condition, and with no other hypothesis. This proves Lefschetz' postulate of singularities, and shows that we may assign any number of cusps up to the maximum, without fear that any case is impossible, i.e., proves our theorem in its entirety.

HARVARD UNIVERSITY

A GENERALIZATION OF NORMAL CONGRUENCES OF CIRCLES*

BY J. L. WALSH

1. *Introduction.* A congruence of circles in three-dimensional space is said to be *normal* if every circle of the congruence is normal to three surfaces. Normal congruences have long been studied,† and one of their principal properties is expressed in the theorem that if a variable circle C is normal to the three fixed surfaces S_1, S_2, S_3 at the points P_1, P_2, P_3 respectively, and if the point P_4 is determined by a real constant cross ratio with P_1, P_2, P_3, then as C varies the point P_4 traces a surface which is also orthogonal to C.

It is the purpose of the present note to consider a type of congruence to which we shall give the name of isogonal congruence and which is a generalization of the notion of normal congruence. A congruence of circles is said to be *isogonal* if every circle of the congruence cuts three surfaces at equal angles in such a way that when the circle is inverted into a straight line L, the tangent planes to the corresponding surfaces at their points of intersection with L are all parallel. That is, every sphere through a circle of the congruence cuts at equal angles the three surfaces at their points of intersection with that circle. It is to be noted that the term isogonal might well be given to a still larger type of congruence of circles, but in the present paper the term will be used only in the restricted sense indicated.

We shall prove (Theorem III) that if a congruence is isogonal there are not merely three surfaces but a one-parameter family of surfaces which have the isogonal property, and all the surfaces of the family can be obtained as in the case of normal congruences.

* Presented to the Society, December 27, 1922.

† By Ribaucour, Darboux, Bianchi, Eisenhart, and Coolidge, among others. Detailed references are given by Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, Chap. XV.

Isogonal congruences of circles are thus a generalization of normal congruences of circles, of normal congruences of lines, and of certain congruences of lines which naturally arise in connection with the parallel mapping of surfaces*. Isogonal congruences are particularly interesting because in general three arbitrary surfaces determine one or several such congruences isogonal to them. If one fixed circle C is isogonal to three surfaces, there is in general one and only one congruence of circles isogonal to those surfaces and containing C. For the condition of isogonality for a circle C' and three surfaces C_1, C_2, C_3 is satisfied if two spheres through the circle C' cut at equal angles the surfaces C_1, C_2, C_3 at their points of intersection with C'. This is equivalent to four independent conditions on all the six-parameter family of circles in space.

Let us proceed to investigate the analogue in the plane of the isogonal congruence.

2. *Isogonal Series in the Plane.* The name *isogonal series* shall be given to a one-parameter family of circles in the plane such that each circle of the family cuts at equal angles three curves, and in such a way that when the circle is inverted into a straight line, the tangents to the transformed curves at their points of intersection with the transformed circle are all parallel. In general, three arbitrary curves determine one or more isogonal series, and if a circle C cuts isogonally three curves there is in general one and only one isogonal series which contains C and all of whose circles cut isogonally those three curves. For the condition of isogonality to three curves involves two independent conditions on the three-parameter family of all circles of the plane. We shall proceed to prove the following theorem.

THEOREM I. *Let a variable circle C cut three fixed curves C_1, C_2, C_3 isogonally at the variable points z_1, z_2, z_3 respectively. Then the point z_4 defined by the real constant cross ratio*

$$(1) \qquad \lambda = (z_1, z_2, z_3, z_4) \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

*Isogonal congruences are also a generalization of a type of congruences of circles considered by F. W. Beal, ANNALS OF MATHEMATICS, (2), vol. 17 (1915–16), pp. 180–170.

traces a curve C_4 such that the circle C cuts isogonally C_1, C_2, C_3, C_4 at the points z_1, z_2, z_3, z_4.

We fix our attention on a particular circle C and its points of intersection z_1', z_2', z_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *if z_1, z_2, z_3 move simultaneously from z_1', z_2', z_3' along C_1, C_2, C_3 in any way whatever, then the point z_4 defined by* (1) *traces a curve which C cuts isogonally with C_1, C_2, C_3.*

If z_1 moves from z_1' along C_1, but z_2 and z_3 are kept coincident with z_2' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_1. This becomes obvious if z_3' is transformed to infinity; equation (1) then represents a transformation (z_1, z_4) of similitude with z_2' as center, while C is a straight line which is unchanged by the transformation. Likewise if z_2 moves from z_2' along C_2, but z_1 and z_3 are kept coincident with z_1' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_2. A similar fact obtains if z_3 moves from z_3' along C_3. Thus, independent infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' move z_4 from z_4' along a curve of the kind described, so simultaneous infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' also move z_4 from z_4' along a curve C_4 such that C cuts the curves C_1, C_2, C_3, C_4 isogonally at z_1', z_2', z_3', z_4'.* This completes the proof of Theorem I.

Theorem I is particularly interesting in the case that C_1,

* The detailed analysis of the differentials involved is extremely simple in the present case. We have by differentiation of (1) and substitution of (1) in the result,

$$(1') \qquad dz_4 = \frac{(z_1-z_2)dz_3+(z_3-z_4)(dz_1-dz_2)-\lambda(z_2-z_3)dz_1-\lambda(z_4-z_1)(dz_2-dz_3)}{(z_1-z_2)+\lambda(z_3-z_3)}.$$

Transform the circle C of Theorem I into the axis of reals, identify the real values z_i' of Theorem I with the z_i of (1'), and identify the variables z_i of Theorem I with the $z_i + dz_i$ of (1'). All the quantities in (1') except the differentials are real; all the differentials of the right-hand member have the same argument (mod π), so dz_4 has also that same argument (mod π).

Thus if $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ are solutions of a Riccati equation whose cross ratio is a real constant, then whenever the $z_i(t)$ vary as functions of t, they trace paths that are cut isogonally by the circle on which the $z_i(t)$ lie.

C_2, C_3 are all circles. In this case we have the following theorem.*

THEOREM II. *Let C_1, C_2, C_3 be three fixed non-coaxial circles. Then the circles C which cut isogonally C_1, C_2, C_3 at points z_1, z_2, z_3 form four distinct series,† each of which is composed of the circles of a coaxial family. If there are considered the circles C of but one isogonal series, the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

traces a circle C_4 which is such that C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one series of circles C cutting them isogonally, namely the circles of the coaxial family conjugate to the family to which C_1, C_2, C_3 belong, and all these circles C cut the three given circles orthogonally. But the points z_1, z_2, z_3 may be chosen on C and on the circles C_1, C_2, C_3 in four essentially different ways, always so that C cuts C_1, C_2, C_3 isogonally at z_1, z_2, z_3. Thus we still have four circles C_4 and for any particular choice of C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts the original circles isogonally at the intersections of C and those circles distinct from P, so no isogonal series is defined. We can still obtain the four circles C_4, however, by requiring respectively (1) that no point z_1, z_2, z_3 shall coincide with P; (2) that z_1 shall always lie at P; C must then be orthogonal to C_1, C_2, C_3; (3), (4) similarly for z_2 and z_3. Always the circle C_4 is traced by the point z_4, and the circle C cuts C_1, C_2, C_3, C_4 isogonally at z_1, z_2, z_3, z_4.

3. *Isogonal Congruences in Space.* Theorem I and its proof

* See Walsh, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 101–116; Lemma IV.

† By a proper convention for the *angle between two circles*, these four systems are described respectively by saying that C cuts C_1, C_2, C_3 all at the same angle or one of those circles at an angle supplementary to the angle cut on the other two. A similar remark obtains below for Theorem IV.

as just given extend directly to space. Let us prove the following theorem.

THEOREM III. *Let a variable circle C cut isogonally three fixed surfaces C_1, C_2, C_3 at the variable points P_1, P_2, P_3. Then the point P_4 defined by the real constant cross ratio*

$$\lambda = (P_1, P_2, P_3, P_4) \tag{2}$$

traces a surface C_4 such that the circle C cuts isogonally C_1, C_2, C_3, C_4 at the points P_1, P_2, P_3, P_4.

We fix our attention on a particular circle C and its points of intersection P_1', P_2', P_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *as P_1, P_2, P_3 move from P_1', P_2', P_3' on C_1, C_2, C_3 in any way whatever, then P_4 as defined by* (2) *traces a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4', where P_4' is defined by $\lambda = (P_1', P_2', P_3', P_4')$.*

If P_1 moves from P_1' along C_1, but P_2 and P_3 are kept coincident with P_2' and P_3', then P_4 traces a surface which is cut by C at P_4' isogonally with C_1, C_2, C_3 at P_1', P_2', P_3'. The corresponding fact holds if P_2 or P_3 is allowed to move on C_2 or C_3 while the other two of the original three points are kept fixed. Independent infinitesimal changes of P_1, P_2, P_3 from P_1', P_2', P_3' along C_1, C_2, C_3 therefore move P_4 along a surface of the kind described, so simultaneous infinitesimal changes of these points must move P_4 along a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4'. Thus even if a congruence is not isogonal but a single circle C of the congruence cuts isogonally the surfaces C_1, C_2, C_3, then C cuts isogonally with C_1, C_2, C_3 the surface traced by the point P_4 defined by (2).

We leave to the reader the proof of the following theorem, which is the space analogue of Theorem II.

THEOREM IV. *Let C_1, C_2, C_3 be three fixed non-coaxial spheres. Then the circles C which cut isogonally C_1, C_2, C_3 at points P_1, P_2, P_3 form four distinct congruences, each of which is composed of the circles through two points, real, coincident, or imaginary. If there are considered the circles C of but one isogonal congruence, the point P_4 defined by the real constant*

cross ratio

$$\lambda = (P_1, P_2, P_3, P_4)$$

traces a sphere C_4 *which is such that* C *cuts isogonally* C_1, C_2, C_3, C_4 *at* P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one congruence of circles cutting them isogonally; all of these circles C cut C_1, C_2, C_3 orthogonally. However, the points P_1, P_2, P_3 may be chosen on C and on their proper spheres in four essentially different ways, and in each case C cuts isogonally C_1, C_2, C_3 at P_1, P_2, P_3. Thus the point P_4 still traces four spheres C_4; and for any particular C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts those spheres isogonally at the intersections of C and these spheres distinct from P, so we have no unique isogonal congruence. There are, however, four spheres C_4 which can be obtained by requiring respectively that P_1, P_2, P_3, or that none of those points should coincide with P. In the former cases the circle C must be orthogonal to C_1, C_2, C_3 to have the proper isogonal property; in the latter case all the circles C to be considered form a complex instead of a congruence. In every case the point P_4 traces a sphere C_4 such that C_1, C_2, C_3, C_4 are cut isogonally by C at P_1, P_2, P_3, P_4.

Theorems III and IV can be extended readily to any number of dimensions.

We add the remark that isogonal congruences arise in space naturally if we consider the problem of finding the locus of a point P_4 defined by the real constant cross ratio

$$\lambda = (P_1, P_2, P_3, P_4) \tag{3}$$

when the points P_1, P_2, P_3 have as their respective loci the regions R_1, R_2, R_3, or the surfaces S_1, S_2, S_3. The surfaces C_1, C_2, C_3 defining the isogonal congruence are the boundaries of the regions R_1, R_2, R_3 or the surfaces S_1, S_2, S_3 themselves. The boundary of the locus of P_4 is traced by P_4 as defined by (3) when the circle C of the congruence cuts C_1, C_2, C_3 isog-

onally at P_1, P_2, P_3. Detailed consideration of the corresponding fact for the plane has been given in a paper by the writer,* and can easily be extended to space by the reader.

HARVARD UNIVERSITY

A CORRECTION

BY EINAR HILLE

In the June number of this BULLETIN (vol. 28, No. 5, p. 261), the author published a paper with the title *Convex distribution of the zeros of Sturm-Liouville functions.* Through an oversight the last paragraph of the paper is inaccurate. We list the necessary corrections below.

Page 264, lines 4–10: Instead of "Note the lineal . . . φ_0.", read "On l we mark the eventual points a_n as well as the points where either arg $G(z) = \arg G(z_1) + \pi$ or $\theta_s \equiv \varphi_0 \pmod{\pi}$. Let $z_2 = z_2(\varphi_0)$ be the first of these points, different from z_1, which we encounter when proceeding along the ray, the rest of which we leave out."

Page 264, line 13: Instead of "an analytic curve", read "either of two analytic curves, namely $A(z_1)$ which is the locus arg $G(z) = \arg G(z_1) + \pi$, and".

Page 264, lines 24–27: Instead of "$l(\varphi_1)$ . . . respectively", read "$l(\varphi_1)$, considered as a double ray if necessary, from z_2^- to z_2 and from z_2 to z_2^+, we make the boundary curve continuous at $\varphi = \varphi_1$".

Page 264, line 28: Instead of "cuts", read "straight lines".

Page 264, line 30: After "the part of", insert "$A(z_1)$ and".

Page 265, first line: Leave out "on the cuts".

Same page, lines 6–10: Replace "Then . . . depends upon z_3" by "Then we can find an angle ϑ such that the two inequalities

$$(15)\qquad \begin{cases} \vartheta < \Theta < \vartheta + \pi; \\ 2k\pi < \Theta < 2(k+1)\pi; \end{cases} \qquad \Theta = \arg\,[G(z)(z - z_1)^2\,],$$

will hold for all interior points on the segment (z_1, z_3), where k is some integer".

PRINCETON UNIVERSITY

* See Lemma III of the paper to which reference has already been made, and also TRANSACTIONS OF THIS SOCIETY, vol. 23 (1922), pp. 67–88, Theorem II.

SHORTER NOTICES

Vorlesungen über Zahlen- und Funktionenlehre. Erster Band, dritte Abteilung. By Alfred Pringsheim. Leipzig and Berlin, B. G. Teubner, 1921. ix + 515–976 pp.

The first part of this volume was reviewed in a previous number of this BULLETIN (vol. 25 (1919), p. 470). The second part, which appeared shortly after the first, was devoted to a detailed exposition of the theory of infinite series with real terms. The third part treats of complex numbers, infinite series with complex terms, infinite products, and continued fractions. It also contains an appendix to the whole first volume and an index. The appendix (pp. 917–969) consists of numerous references to the literature and indications of the historical development of portions of the theory, as well as further discussion of many of the topics treated in the body of the text.

The exposition exhibits Professor Pringsheim's characteristic lucidity, and the theory throughout is developed from first principles in elementary fashion. By the phrase "elementary fashion" we do not wish to connote anything in the nature of looseness of treatment or lack of rigor, for the author has been at particular pains to avoid defects of that sort. Thus the book is well adapted to meet the needs of any reader, regardless of the extent of his previous mathematical knowledge, who wishes to have a complete and logically accurate account of the arithmetic foundations of modern analysis.

In spite of the elementary treatment, the book is in no way limited to the elements of the subject and in connection with various particular topics comes close to the confines of our present-day mathematical knowledge. For example in the second chapter of Section III,* as a preparation for the treatment of the multiplication of series, the methods of Hölder and Cesàro for summing divergent series are introduced and the equivalence of these two methods, a theorem of comparatively recent date, is established. In this connection it is of interest to note that Professor Pringsheim objects to the use of the phraseology "summation of a divergent series" and introduces in place of it the expression "reduction (Reduktion) of a divergent series." He thus speaks of a certain series as being reducible (reduzibel) with a particular associated limit (zugeordneter Grenzwert). He bases his objection to the current usage on the statement that at the present time there exists no precise definition of the term "sum of a divergent series," and in his opinion an epithet so expressive and sonorous (ein so prägnant klingendes Beiwort) as summable should have a precise and definite meaning, such as the words convergent and divergent possess.

It will be recalled by those particularly interested in the study of divergent series that the objections raised by Professor Pringsheim to the present use of the word summable are quite similar to those previously expressed by Professor W. B. Ford in his book *Studies on Divergent*

* The whole first volume has been divided into four principal sections, of which Part three contains Sections III and IV.

Series and Summability and his address entitled *A conspectus of the modern theory of divergent series*, published in volume 25 of this BULLETIN. It will also be recalled that the remedy proposed by the latter was more drastic, namely to give the word summable a definite meaning, based on the notion of analytic extension, and thus exclude from consideration any divergent series that did not satisfy the proposed definition of summability. A combination of the suggestions of Professors Ford and Pringsheim, namely the use of Professor Ford's definition to set off a definite class of divergent series to be known as summable series, and the use of Professor Pringsheim's term reducible in connection with other divergent series might be more generally acceptable than either suggestion alone. It is the opinion of the reviewer, however, that the present use of the word summable, even if it does involve a certain vagueness in the meaning of such an excellent word, is rather too firmly entrenched to be changed overnight. But the objections of Professors Ford and Pringsheim do have a certain validity, and regardless of the opinion of particular individuals the final terminology will probably be an illustration of the survival of the fittest.

In connection with the introduction of complex numbers in Chapter I of Section III, the author departs from the generally adopted method of first building up a system of pairs of real numbers with arbitrarily assigned rules of combination. He begins by introducing the pure imaginary as a new type of number adapted to provide a solution for equations of the type $x^2 = -a^2$ (a real). Then after developing the properties of this new system on the basis of the principle that in their rules of combination they should obey the fundamental laws of algebra, he leads naturally to the introduction of complex numbers by showing that the result of adding a real number and an imaginary number will not be a number of either of these types.

The treatment of continued fractions, to which all of Section IV is devoted, is particularly complete. As regards questions of convergence it contains practically all of our present-day knowledge of the subject. In his preface the author expresses the hope that this part of the book will serve to gain for this highly important, and in his opinion not sufficiently appreciated, topic new devotees from the ranks of mathematical novitiates.

The book proper contains relatively few references to the literature, these having been in the main relegated to the appendix. This arrangement is in line with the purpose of making the treatment particularly suitable for beginners, since it avoids undue interruption in the orderly presentation of the theory and the possibly forbidding aspect of numerous footnotes and citations in the body of the text. With regard to the appendix the author specifically disclaims any pretensions to completeness and excuses himself in advance in case he may on this account have passed over priority claims of particular individuals. In this connection he expresses the opinion that the custom of attaching the names of the discoverers to mathematical theorems has been carried too far in recent times and has led to what he considers a depreciation of the value of mathematical immortality. For this reason he has in general avoided the use of the name of the originator in connection with a theorem or a method except

in instances where the usage may be regarded as classical. One is tempted to inquire, however, how these classical usages could have originated if some one had not initially honored an author by naming one of his discoveries after him? It would seem that the whole question reduces to a matter of taste, as to whether one prefers the personal or the impersonal point of view in the study of mathematical science.

C. N. MOORE

Theorie des Potentials und der Kugelfunktionen. By A. Wangerin. Band II, Sammlung Schubert LIX. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. viii + 286 pp.

The first volume of the above work, which deals with the fundamental portions of the potential theory, was reviewed in a previous number of the BULLETIN (vol. 16 (1910), p. 492). The present volume is devoted to a study of the properties of spherical harmonics and their applications to various problems of potential theory.

The book is divided into four sections. The first section deals with such properties of spherical harmonics as are essential for later developments; the second deals with potential problems for the sphere; the third with potential problems for the ellipsoid of revolution, for two spheres, and for a few other special cases; the fourth with potential problems for arbitrary closed surfaces.

The book as a whole contains more material than one would expect to find in a work of its size, and the presentation is in general clear and sufficiently refined for the purpose in hand. There are, however, certain errors and inconsistencies in the statements and demonstrations of some of the fundamental theorems that should be eliminated. For example, any one familiar with the literature on spherical harmonics will be surprised to find on page 93 the claim that the author has established the convergence of the development in Laplace's functions of any function $f(\theta, \varphi)$ that is finite, single-valued, and continuous on the whole sphere. Since examples have been constructed by Haar and Gronwall of continuous functions whose development in spherical harmonics is divergent, one naturally examines the "proof" of the so-called theorem with some curiosity. On page 89 he finds the assumption that a sum of p terms, each one of which approaches zero, will also approach zero, even though p becomes infinite at the same time that the individual terms approach zero. On page 93 he finds a statement that is equivalent to the assumption that a function continuous throughout a certain interval does not have an infinite number of maxima and minima in that interval.

It would surely be better to omit entirely any proof of the development theorem, important as it is for the subsequent theory, than to introduce such obvious errors with regard to the fundamentals of analysis in the course of the demonstration. A less radical remedy is available, however, for by adding a further restrictive condition to the statement of the theorem and a relatively small amount of material to the proof, the whole discussion could be put on an entirely rigorous basis. It is to be hoped that these changes will be made, if a subsequent edition appears.

C. N. MOORE

Lezioni sulla Teoria dei Numeri algebrici e Principi d'Aritmetica analitica. By Luigi Bianchi. Pisa, Enrico Spoerri, 1921. viii + 444 pp.

This work consists of a 50-page introduction and three chapters. The introduction contains a development of the theory of numbers for the field $k(i)$. The theorems of the theory of the rational numbers are extended to this field giving the proof of the unique separation into prime factors, the development of the φ function, the proof of Fermat's theorem, the theory of binomial congruences with proof of the existence of primitive roots of the congruence $x^{\varphi(p)} \equiv 1 \bmod p$, and the theory of indices and quadratic residues with the law of reciprocity.

The first and second chapters are devoted to the theory of algebraic numbers in a general field of degree n. The first chapter contains the development of that part of the theory which is independent of factorization. The existence of a base is shown and the discriminant of a field is defined. The greater part of the chapter is devoted to the theory of units which is developed from Minkowski's theorem on linear forms leading up to the proof of the existence of a fundamental system of units.

The second chapter contains the theory of ideals, showing the purpose of their introduction into the theory of algebraic numbers, proving the unique factorization theorem, and discussing congruences with respect to an ideal modulus. The chapter also contains the theory of the classification of ideals, the discussion of the group of classes, and the correspondence between the classes of ideals and equivalent decomposable forms.

The last part of the chapter is given over to a brief consideration of the theory of ideals in Galois, Abelian, cyclotomic, and Kummer fields. The law governing the factorization of the rational primes in a cyclotomic field of prime degree is developed. This is also done earlier in the chapter for quadratic fields as an application of the theory of quadratic residues with respect to an ideal modulus.

The last chapter contains a good presentation of the analytic theory of ideals. The Riemann and Dedekind zeta functions are studied showing the relation of the latter to the problem of the determination of the number of classes in a field.

The zeta functions for quadratic and cyclotomic fields are reduced to their simpler forms and the method for the determination of the number of classes in a quadratic field is given for the case when the discriminant of the field is congruent to 1 mod 4.

The last paragraph contains a brief indication of Hecke's results regarding the zeta function.

G. E. WAHLIN

Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale. By Dr. Edmund Landau. Leipzig and Berlin, B. G. Teubner, 1918. viii + 143 pp.

This work is divided in two parts. The first part (54 pages) contains a brief, but unusually clear, presentation of the elementary theory of algebraic numbers and ideals in an algebraic number field.

The author states in the preface that the aim of the work is not to give

an exhaustive treatment, but to bring the reader up to and a little beyond the fundamental theorem of Dedekind that the separation of an ideal into the product of prime ideals is unique.

The subjects considered beyond the work necessary for the proof of the theorem mentioned are: the theory of the classification of ideals showing that the number of classes is finite, and the theory of units in an algebraic number field, with the proof of the existence of a fundamental system of units.

The second part is devoted to the analytic theory of ideals. The first chapter gives a clear introduction to the functions $\zeta(s)$ and $\zeta(s;\kappa)$, where κ is a class of the field. It is shown that, except for a pole of the first order in $s = 1$, the functions are regular in the entire plane, and that the residue of $\zeta(s;\kappa)$ in $s = 1$ is a number λ, independent of the class κ, while that of $\zeta(s)$ is λh, where h is the number of classes. Hecke's functional equations for $\zeta(s)$ and $\zeta(s;\kappa)$ are also developed.

The second chapter is a study of the distribution of the zeros of $\xi(s)$, and the third chapter leads to the proof of the author's remarkable theorem that, asymptotically, the number of prime ideals is the same in all fields.

The last chapter contains the result of the author's researches regarding the number of ideals, in a field or a class, whose norms are less than x. If this number be denoted by $H(x)$ for the field and $H(x;\kappa)$ for a class, it is shown that

$$H(x;\kappa) = \lambda x + O(x^{\vartheta})$$
$$H(x) = \lambda h x + O(x^{\vartheta})$$

where λ and h have the meaning given above,

$$\vartheta = 1 - \frac{2}{n+1}$$

and $O(x^{\vartheta})$ is a function whose quotient by x^{ϑ} is limited for sufficiently large x. It is shown that the exponent ϑ cannot be less than

$$\frac{1}{2} - \frac{1}{2n}.$$

The last four pages contain a brief historical survey with examples from a quadratic field, as well as notes of reference to the literature bearing on the various sections.

G. E. WAHLIN

Lehrbuch der Funktionentheorie. By L. Bieberbach. Band I. *Elemente der Funktionentheorie.* Leipzig, Teubner, 1921. 6 + 314 pp.

Funktionentheorie. By L. Bieberbach. Teubner's Technische Leitfaden, Bd. 14. Leipzig, Teubner, 1922. 118 pp.

The author in his preface to the first book pleads guilty to entertaining the hope that he has written a text-book on complex function theory. He proceeds to set forth the qualities that such a text-book should have, viz: completeness, clarity, simplicity and unity. That the author succeeds in giving a clear, elementary presentation of the fundamental principles of the theory of functions of a complex variable cannot be gainsaid.

The table of contents indicates the usual order of topics.

1. Elementary treatment of complex numbers as number pairs, treated arithmetically and geometrically. 2. Limits and series, containing the fundamental concepts and definitions. 3. Continuity, region, derivatives, series of functions, conformal mapping. 4. The elementary functions of a complex variable. 5. Integration. 6. Cauchy's integral formula. 7. Theory of residues. 8. Analytic continuation. 9 and 10. Algebraic functions. 11. Elliptic functions. 12. Simple periodic functions. 13. Expansion of analytic functions in infinite series and products. 14. The Gamma function.

The printing is very good, unusually free from errors. The reviewer wishes to point out that on page 34 in the paragraph on the inverse function defined by $w = f(z)$ the requirement $f'(z_0) \neq 0$ should be added. Again on page 35 in the paragraph in which Laplace's differential equation is derived, the condition necessary to make the equation $v_{xy} = v_{yx}$ valid is not stated.

The eighty figures in the book are illuminating and helpful. Perhaps the chief contribution is contained in the use of the modern point-set theory in the fundamental definitions. This is a logical development of the fundamental importance which this theory is rapidly assuming in all branches of mathematics, especially in function theory. Two important theorems are to be noted, the first an extension of Weierstrass's theorems on a uniformly convergent series of analytic functions and an extension of Mittag-Leffler's theorem. The first, on page 165, is due to Vitali, and the second, on page 292, is due to Runge. The book as a whole is a welcome addition to complex function theory and marks a distinct contribution.

The second book is a simplified summary of the Lehrbuch and possesses the same merits as the larger text. The last two chapters contain applications to potential theory and hydrodynamics.

H. J. ETTLINGER

Vorlesungen über Höhere Mathematik. By Hermann Rothe. Vienna, L. W. Seidel und Sohn, 1921. xi + 691 pp.

This book is strikingly like the first volume of Pierpont's *Theory of Functions of a Real Variable*, both in scope and in the spirit of rigor which pervades it. Its larger, more closely printed pages contain much more detail of proof, but beyond a considerable wealth of illustrative examples and some geometric applications of the calculus, the material and the degree of generality of the theorems are almost identical. At many points, the reader will find the proofs easier to study than those of Pierpont because every step is supplied. But at other points, he will find this same trait wearying and obscuring to the real essence of the demonstration. It is not likely that the author had any knowledge of the *Theory of Functions of a Real Variable*, for in supposing that he might be offering the first valid proof of the theorem on the derivative of a function of a function (see the preface), the only place in which he claims novelty of results, he overlooks the satisfactory proof given in that book.

The fact that these *Vorlesungen* are an elaboration of lectures given in a prescribed course in the Technische Hochschule at Vienna will mislead

no one in this country into thinking that they can be of large service to technical students here. We appreciate rigor, to be sure, and feel the need of more of it. But we want also some analytical skill, we want some treatment of series, including Fourier series, and of approximation processes, and we want practice in the application of the calculus to problems of physics and engineering. And we do not care particularly for unusual generality at the expense of simplicity of proofs.

For the student of pure mathematics, on the other hand, the book has value. He may consult it with confidence, and will find it useful in supplying alternative proofs, or in supplying details of proofs given too succinctly elsewhere. He will find it well printed, well arranged, and supplied with figures which are models of clearness, if exception be made of a few, which, like the text, are overloaded with detail.

It is regrettable that so many mathematical texts are written without any clear purpose of stimulating self-activity. In the *Vorlesungen* there is not a single problem to be worked, and no detail of proof to be supplied. Many a mathematician owes his interest in the science to early bouts with problems, and the writer who gives careful attention to the selection of problems, or who includes as exercises valuable theoretical results, renders a high service. Both of these merits characterize, for instance, Appell's *Mécanique Rationnelle*, and Goursat's *Cours d'Analyse*. Had our present author followed some such course, he might have saved space, and produced a much more stimulating book.

O. D. KELLOGG

Plane Geometry. By L. B. Benny. Glasgow and Bombay, Blackie and Son, Ltd., 1922. vi + 336 pp.

The title of this book is misleading to American readers. The sub-title is as follows: "An account of the more elementary properties of conic sections treated by the methods of coordinate geometry and of modern projective geometry with applications to practical drawing." The contents will appeal to the American reader as a rather unusual mixture of elementary analytic geometry and projective geometry. In addition to the ordinary analytic treatment of the straight line and conic sections, the book contains chapters on ranges and pencils, harmonic properties of circles, inversion, projection, confocal conics, cross ratios and ends with a treatment of Pascal's and Brianchon's theorems. While the selection of contents is an unusual one in this country, the reader will find much of interest. Like most English texts, it is well supplied with exercises and problems of a type and difficulty unusual on this side of the Atlantic. The book also contains a number of portraits of mathematicians who have contributed to the subjects under discussion with an appendix giving biographical notes concerning them. These include Cayley, Riemann, Cremona, Descartes and Pascal.

J. W. YOUNG

Cours de Géométrie Analytique. By Georges Bouligand, with a preface by E. Cartan. Paris, Librairie Vuibert, 1919. vii + 421 pp.

This book constitutes a second course in the Cartesian geometry of the plane and of space, freely interspersed with essentials of modern geometry. After a concise development of rectangular and oblique coordinates and their application to straight lines and planes, the infinite and the imaginary elements are introduced. With the basis for projective geometry in complex space or for any of its sub-geometries thus developed, the author proceeds to consider some general questions. He first treats miscellaneous elementary properties of real curves and surfaces in real space. He then passes on to the discussion of projective and affine properties of algebraic curves and surfaces in complex space, developing the ideas of the order of a curve or a surface and that of one of its points, proving Bezout's theorem, and devoting some attention to asymptotes and to unicursal curves. After a brief discussion of loci comes the next major topic, that of transformations. Rigid motions, reflections, and transformations of similitude are discussed, followed by affine and projective transformations; cross ratio is introduced and the projective generation of conics and quadrics developed, involutions are given proper attention and Desargues' theorem is proved; finally comes the geometry of inversion.

An important phase of this chapter is the stress laid on the distinction between metric, affine, and projective geometries. Indeed, if the reviewer were to single out any one motive governing the plan of the book, it would be this motive. Latent but still decisive in the beginning, when once introduced it maintains predominance.

One further general concept, that of tangential coordinates, is developed, and the author then attacks the theory of curves and surfaces of the second order. With the general equation as the point of departure in each case, the usual questions of classification, normal forms, various geometric definitions (of the conics), rulings and circular sections (of the quadrics), tangential equations, poles and polars, diameters, etc., are discussed; a separate chapter treats geometrically the intersection of two quadrics.

A supplement of some sixty pages covers the geometrical applications of determinants, the classification of conics and quadrics on an analytic basis, the determination of conics and quadrics under given conditions, pencils of conics and quadrics, and an introduction to the theory of invariants.

The differential calculus is freely applied and a modicum of differential geometry developed. Certain subjects, whose detailed analytic treatments are regarded, perhaps unfortunately, by most American geometers as belonging to analysis, are carefully treated: the implicit function theorem in connection with a plane curve defined by an implicit equation, and a thorough analytic theory of envelopes in connection with line coordinates. On the other hand, the principle of duality is barely mentioned and, though polar reciprocation with respect to the sphere, $x^2 + y^2 + z^2 + 1 = 0$, is freely used, the theory of reciprocal polars in general is lightly passed over and that of general correlations not mentioned.

All in all, however, the author is to be congratulated on his choice of

material; only essentially important topics are treated and these carefully and fully. There are no exercises per se, but the author leaves much in the text to the student and furnishes frequent suggestions for further study. The methods followed are for the most part analytic, but are inspired by a keen geometric insight. The proofs and deductions are concise, rigorous, facile, and in some cases, novel.

The author has followed an established custom in the descriptive, rather than logical, introduction of imaginary elements. This has led here, as frequently, to a lack of proper respect for the eccentricities of the complex domain, resulting in some unguarded, if not inaccurate, statements. It is to be hoped that the time is not far distant, when geometers who feel themselves forced into the complex domain will accord to it a logical and rigorous foundation.

The book makes delightful reading. It should prove of value to advanced students and to teachers of Cartesian and modern geometry in this country.

W. C. GRAUSTEIN

Applied Calculus. By F. F. P. Bisacre. Glasgow and Bombay, Blackie and Son, Ltd., 1921. xiv + 446 pp.

It is not in general the policy of the BULLETIN to review elementary textbooks on the calculus. In the present instance, however, it seems desirable to make an exception and to give a brief notice to Mr. Bisacre's text. It differs in several particulars from the usual elementary textbook. Its avowed intention is to provide an introductory course in the calculus for the use of students in the natural and applied science whose knowledge of mathematics is slight. The author does not presuppose anything beyond elementary algebra, trigonometry, geometry and graphs. In particular no previous course in analytic geometry is presupposed. The first part of the text as far as its selection of material is concerned does not differ very greatly from the ordinary textbook. It is, however, distinguished by a remarkable vividness of presentation. The author uses a quotation from the *Merchant of Venice*, for example, to illustrate the notion of units and large and small quantities, in particular the notion of allowable error, etc. In the latter part of the text, the selection of material, however, differs widely from the conventional. There is a chapter of thirty-two pages on problems in electricity and magnetism, a chapter of twenty-one pages on problems in chemical dynamics and a chapter of sixty-five pages on problems in thermodynamics. The treatment is remarkably accurate and rigorous considering its elementary character. Its interest is further enhanced by biographical notes and portraits of mathematicians who have contributed to the development of the calculus and to the physical theories covered.

It is doubtful whether the book could be advantageously used in courses on the calculus in this country and yet it is not at all clear that some schools of applied science or technology might not find it distinctly available in their work.

J. W. YOUNG

Die Grundgleichungen der Mechanik, insbesondere starren Körper. Neuentwickelt mit Grassmanns Punktrechnung. By A. Lotze. Leipzig, B. G. Teubner, 1922. 50 pp.

A great many quantities which enter into mechanics are vectors and consequently the most natural way to treat mechanics is by vector methods and this has been done by a great many writers. There are, however, difficulties in treating forces which act at a given point, for vectors in general are only determined in magnitude and direction and hence to locate them on definite lines brings in other considerations.

The Grassmann point analysis gives us, however, a natural way out of this difficulty. He considered two elements, A–B (where A and B are points) which represents a vector in the ordinary sense of the word, and AB which represents the segment of the line joining the points A and B. In cases, then, when we wish to localize a vector we can indicate it by AB.

In this little pamphlet Lotze writes up quite an extensive treatment of mechanics from the point of view of Grassmann's analysis. He assumes a knowledge of the point analysis including the notions of the Lückenausdruck and the fraction. No discussion of this is given and in places the argument is not easy to follow. The author has introduced some symbols of his own or at least not known to the reviewer, e.g., in addition to Grassmann's complement he uses $\llcorner\ \bar{v}$ to indicate the vector, in a plane, into which $\bar{v}$ rotates by a positive rotation through $\pi/2$; $\perp\ \bar{v}$ indicates (in space) the 2-vector perpendicular to $\bar{v}$ and of equal magnitude and so directed that $\bar{v} \perp \bar{v} = v^2$. Different symbols are used to represent the quantities of different order and this lessens the difficulty of reading.

This is a fairly complete text of the mechanics of rigid bodies. It is divided into three chapters: I. Kinematics of rigid bodies; II. General dynamics of material point systems; III. Dynamics of rigid bodies. The general properties of rigid motion are quite fully treated in the first chapter. The second chapter carries us as far as the derivation of d'Alembert's and Hamilton's principles and Lagrange's equations. The last chapter deals with work and energy and the various screws such as the impulse screw and the force screw.

The pamphlet is well worth reading; but it seems to the reviewer as if the reading could have been made much easier.

C. L. E. Moore

Précis d'Arithmétique. By J. Poirée. Paris, Gauthier-Villars et Cie., 1921. 62 pp.

C. Camichel has written a preface for this delightful little volume in which he says, "L'Arithmétique élémentaire est une excellente introduction à l'étude des Mathématiques. On y trouve sous une forme concrète des modèles de tous les modes de raisonnement depuis les plus simples jusqu'aux plus délicats de l'Analyse. Cependant cette partie des Mathématiques est en général négligée par les élèves." Poirée has presented a few topics from the theory of arithmetic and the theory of numbers in a way that will attract the neophyte and will be approved by the savant. The discussion commences with "Combien y a-t-il de billes?" and leads up to

"Recherche des Racines primitives d'un Nombre premier p." The idea of limit is introduced in connection with fractions, irrationals are connected with square root, logarithms with progressions. Classical theorems and concepts are introduced throughout the book: Euclid's proof of the infinitude of primes, sieve of Eratosthenes, number of divisors and sum of divisors of an integer, Cauchy's indicator, elementary theory of the congruence, Fermat's minor theorem, Wilson's theorem.

Possibly a book of this type needs no references, although some would probably lead a few readers to wider study. The book contains numerous good examples, but is without problems or stimulating questions for its reader. It is an engaging monograph with hardly a typographical flaw, and the reviewer believes that it will be of service either to organize and clarify the mathematical thinking of the younger or to direct students into number theory.

L. C. MATHEWSON

Latitude Developments Connected with Geodesy and Cartography. By Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1921. Special Publication No. 67. 132 pp.

Elements of Map Projection. By Charles H. Deetz and Oscar S. Adams. Washington, United States Coast and Geodetic Survey, 1921. Special Publication No. 68. 160 pp.

The first of these little books discusses the various "kinds of latitude" that arise in questions connected with geodesy and cartography. Five of these are discussed; namely, 1. *Geodetic, or astronomical, latitude* is the angle which the normal to the earth's surface at any point makes with the major axis of the meridian ellipse through the point; 2. *Geocentric latitude* is the angle which the radius vector makes with the major axis; 3. *Parametric latitude* is the parametric angle when the equation of the meridian ellipse is written in the usual parametric form; 4. *Isometric latitude* is connected with the conformal representation of the earth upon a sphere; 5. *Authalic latitude* is connected with the "equal area" representation of the earth upon a sphere (authalic from Tissot).

The book deals with formulas giving the last four in terms of the first and the eccentricity of a meridian. The formulas are applied to the Clarke spheroid of 1866, and there is a complete table for changing from one latitude to another.

The second book, as its title indicates, is devoted to various methods for constructing maps of the earth's surface, to a discussion of the advantages and disadvantages of each, and to the relative distortions introduced by each. The first half of the book is mostly descriptive and is geometrical in character. The latter half contains a full development of the Mercator projection with tables for computation, and also a mathematical discussion of various other projections. There is an explanation of the French "grid system" so much in use during the war.

The book has many excellent diagrams and maps and can be read by any one familiar with the calculus.

L. W. DOWLING

Lessen over de Hoogere Algebra. Vol. I. By Fred. Schuh. Groningen, P. Noordhof, 1920. xx + 526 pp.

This work, which is to be complete in two volumes, was planned to be a ninth edition of Lobatto's *Higher Algebra* and on its title page is being announced as such. But although designed at its inception to be simply a revision of the eighth edition of Lobatto, the changes and additions which have arisen are so numerous as to warrant the appearance of these volumes as texts by Dr. Schuh himself. Very little has been taken bodily from Lobatto's text, the material taken over unchanged being primarily the examples and exercises used; but even with respect to these much new material has been added. These additions have so enlarged the work as to necessitate its publication in two volumes.

In volume I, the new material includes the treatment of theory of linear transformations and quadratic forms and the closely allied characteristic equation, called by Dr. Schuh the "secular equation" or "S-equation." To compensate for the quantity of additional material, there has been eliminated from the Lobatto text the treatment of permutations and combinations as they relate to the binomial theorem and to powers of polynomials, the reader being referred for these subjects to the secondary algebra published by Dr. Schuh in collaboration with P. Wydenes, which publication is to be considered as the connecting link between Wydenes' *Elementary Algebra* and the present two volumes of Dr. Schuh. Following the chapters dealing with determinants, linear equations and transformations, quadratic forms, simple properties of polynomials in x, are chapters in which D'Alembert's, Rolle's, Des Cartes' Budan-Fourier's and Sturm's theorems and their applications are treated in full. The concluding chapter treats the characteristic equation. The work is replete with examples, there being 815 in this first volume. The text is an admirable one suited to an advanced undergraduate course in college algebra.

J. N. Van der Vries

Encyklopädie der Mathematischen Wissenschaften. Volume II, Part II. Edited by H. Burkhardt, W. Wirtinger, R. Fricke, and E. Hilb. Leipzig, B. G. Teubner, 1901–1921. xv + 897 pages.

This part of the German encyclopedia of mathematics contains articles by Osgood on the general theory of analytic functions of one and of several complex variables, by Wirtinger on algebraic functions and their integrals, by Fricke on elliptic functions and on automorphic functions including elliptic modular functions, by Hilb on linear differential equations in the complex domain and on non-linear differential equations, by Krazer and Wirtinger on Abelian functions and general theta functions. The first article was completed in 1901 and the last in 1920. The authors, editors, and publishers are to be congratulated on the completion of this part of the encyclopedia. It will be of the greatest service to the students of analysis. Indeed, the earlier articles have been in use for many years. Americans are proud that the article on analytic functions is written by a noted American mathematician who has contributed so much to the field.

A. D. Pitcher

Funktionentheoretische Vorlesungen. By H. Burkhardt. Berlin, Vereinigung Wissenschaftlicher Verleger. Band 1. Heft 1. *Algebraische Analysis.* Dritte umgearbeitete Auflage, besorgt von G. Faber, 1920. 10 + 182 pp. Band 1. Heft 2. *Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen.* Fünfte umgearbeitete Auflage, besorgt von G. Faber. 1921. 10 + 286 pp.

The first edition of this excellent and well known elementary treatment of function theory appeared in 1897. Since then it has appeared in three further editions, in 1903, 1907, and 1912 respectively, under the revision of the author. The latter died in 1914. This fifth edition of the second section and the third edition of the first section have been issued under the direction of Dr. G. Faber who succeeded Burkhardt at the Technische Hochschule in Munich. The first edition of Heft 2 was carefully reviewed by M. Bôcher in this BULLETIN, vol. 5 (1898–99), pp. 181–185. The second edition of Heft 2, together with the first edition of 1, was reviewed by L. E. Dickson in this BULLETIN, vol. 10 (1903–04), pp. 317–321. The third and fourth editions present no essential changes. It is to be noted that S. E. Rasor has provided a good English translation of the fourth edition and has supplied splendid sets of problems to accompany each chapter. The translation is published by D. C. Heath and Co.

In the present edition no fundamental changes appear, and the method and spirit of Burkhardt have been retained by Faber. Whatever slight changes have been made are in the interest of clarity and simplicity. These two books should continue to furnish an excellent introduction to the theory of functions, both real and complex.

H. J. ETTLINGER

Géométrie Perspective. By M. Emanaud. Paris, Octave Doin, 1921. xv + 432 pp.

The Encyclopédie Scientifique published under the direction of Dr. Toulouse contains as a sub-class the Bibliothèque de Mathématiques Appliquées edited by M. d'Ocagne of nomographic fame. The volume under review belongs to this series and has been written by M. Emanaud, the chief of graphic courses at the École polytechnique. In an introduction of 39 pages the author gives a brief account of the properties of collineations in two and three dimensions with special reference to their applications in perspective constructions. This is a very commendable feature of the book. In fact, a scientific base of this sort is indispensable for a successful and up-to-date course in perspective. When years ago the reviewer was teaching artistic perspective at the cantonal technicum in Biel, Switzerland, he followed the same method.

M. Emanaud's book is very ably written. It covers the usual topics in perspective. Of particular interest are the chapters on perspective in art, on theatrical perspective, and on the construction of dioramas and panoramas. The only serious criticism which the reviewer has to make pertains to the figures, which are very poorly drawn and lettered.

A. EMCH

NOTES

At the meeting of the Chicago Section of this Society on April 13–14 in Chicago, Professor S. Lefschetz, of the University of Kansas, will deliver a symposium lecture on *Curves traced on algebraic surfaces.*

The opening number of volume 23 of the TRANSACTIONS OF THIS SOCIETY contains the following papers: *Relatively uniform convergence and the classification of functions*, by E. W. Chittenden; *Periodic functions with a multiplication theorem*, by J. F. Ritt; *Note on Dirichlet and factorial series*, by Tomlinson Fort; *Functions of infinitely many variables in Hilbert space*, by W. L. Hart; *Prime and composite polynomials*, by J. F. Ritt; *Some two-dimensional loci connected with cross ratios*, by J. L. Walsh; *On transformations with invariant points*, by J. W. Alexander; *Invariant points in function space*, by G. D. Birkhoff and O. D. Kellogg. Volume 23 is the extra volume presented to the Society by an anonymous donor (see this BULLETIN, vol. 28, p. 234).

The opening number of volume 44 (1922) of the AMERICAN JOURNAL OF MATHEMATICS contains: *An arithmetical dual of Kummer's quartic surface*, by E. T. Bell; *Incidences of straight lines and plane algebraic curves and surfaces generated by them*, by Arnold Emch; *On the theorems of Gauss and Green*, by V. C. Poor; *An extension of the Sturm-Liouville expansion*, by C. C. Camp; *Conformal transformations of period n and groups generated by them*, by Harry Langman.

The December, 1921, number (vol. 23, No. 2) of the ANNALS OF MATHEMATICS contains: *Transformations of trajectories on a surface*, by Joseph Lipka; *On the structure of finite continuous groups with one two-parameter invariant subgroup*, by S. D. Zeldin; *On the simplification of the structure of finite continuous groups with more than one two-parameter invariant subgroup*, by S. D. Zeldin; *The automorphic transformation of a bilinear form*, by J. H. M. Wedderburn; *A direct determination of the minimum area between a curve and its caustic*, by Otto Dunkel; *The Poisson integral and an analytic function on its circle of convergence*, by A. Arwin; *Systems of circuits on two-dimensional manifolds*, by H. R. Brahana; *Two generalizations of the Stieltjes integral*, by P. J. Daniell.

Two volumes of the PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE DE L'UNIVERSITÉ DE STRASBOURG have appeared, in 1920 and 1922. They contain memoirs published by the members of the faculty of the University.

At the meeting of the National Academy of Sciences held in New York City November 14–16, 1922, the following mathematical papers were read: *On the attraction of a central body in the theory of relativity*, by G. D. Birkhoff; *The equiaffine geometry of paths*, by Oswald Veblen; *The cosmological equation of gravitation*, by Edward Kasner; *The use of isophelimatic lines in*

historical geography, A new theory of population, A new hydrodynamica phenomenon, and *Further results in interior ballistics,* by A. G. Webster *Affine geometry of paths possessing an invariant integral,* by L. P. Eisenhart. The Draper Gold Medal was presented to Professor H. N. Russell, of Princeton University.

At the meeting of the International Research Council at Brussels, in July, 1922, the following executive committee was elected to serve until 1925: E. Picard, president; G. Lecointe and V. Volterra, vice-presidents; Sir A. Schuster, secretary general.

The following awards of Nobel prizes in physics have been announced: for 1921, to Professor Albert Einstein, for his theory of relativity and his general work in physics; for 1922, to Professor Neils Bohr, for his researches in the structure of atoms and radiation.

The Paris Academy of Sciences has awarded its Lalande Medal to Professor H. N. Russell, of Princeton University, and its Janssen Medal to Dr. Carl Stormer, professor of pure mathematics at the University of Christiania, for his work on the aurora borealis.

Dr. J. G. Hagen, director of the observatory of the Vatican, has been elected non-resident member of the Accademia dei Lincei.

The Royal Society of London has awarded its Sylvester Medal to Professor Tullio Levi-Civita, for his researches in geometry and mechanics.

The Franklin Medal of the Franklin Institute of Philadelphia has been presented to Sir J. J. Thomson.

Professor J. H. Jeans received the doctorate of science from Oxford University on the occasion of his delivery of the Halley lecture.

Professor N. E. Nörlund, of the University of Lund, has been called to the University of Copenhagen.

Professor F. von Dalwigk, of the University of Marburg, has been called to the Potsdam Geodetic Institute.

Dr. H. König, of the University of Göttingen, has been appointed professor of mathematics at the school of mines at Clausthal.

Dr. E. Kruppa has been appointed associate professor of mathematics at the Vienna Technical School.

Professor H. von Sanden, of the school of mines at Clausthal, has been appointed to a professorship at the Hannover Technical School.

Dr. C. Siegel, of the University of Göttingen, has been appointed professor of mathematics at the University of Frankfurt am Main.

Professor R. Mehmke, of the Stuttgart Technical School, has retired from active teaching.

Dr. A. Ostrowski has been admitted as privat docent at the University of Hamburg.

Dr. Janet has been appointed professor of pure mathematics at the University of Rennes.

Professor S. K. Banerji, of the department of applied mathematics at the University of Calcutta, has resigned to become director of the observatories at Bombay.

The Council of the University of Sheffield has appointed Professor A. H. Leahy emeritus professor of mathematics.

Mr. G. P. Thomson, lecturer in mathematics at Corpus Christi College, Cambridge, has been appointed professor of natural philosophy at the University of Aberdeen, as successor to Professor C. Niven, retired.

Mr. E. C. Francis, of Trinity College, has been elected fellow and lecturer in mathematics at Peterhouse, Cambridge.

Associate Professor C. F. Gummer has been promoted to a full professorship at Queen's University, Kingston.

Professor A. Sommerfeld will deliver a course of lectures at the Bureau of Standards, Washington, early in March, 1923, on the quantum theory and related subjects.

Associate Professor C. M. Sparrow, of the University of Virginia, has been promoted to a full professorship of physics.

Dr. T. H. Brown, of Yale University, has been appointed assistant professor of foreign trade at Columbia University.

Professor Glenn James, of the Carnegie Institute of Technology, has been appointed assistant professor of mathematics at the southern branch of the University of California.

Mr. R. M. Deming, of the Case School of Applied Science, has been appointed professor of mathematics at Upper Iowa University, Fayette.

Mr. F. H. Murray has been appointed instructor in mathematics at the University of West Virginia.

The death is reported of Professor A. V. Bäcklund, of Lund, at the age of seventy-seven years.

Professor C. A. Fischer, of Trinity College, Hartford, died December 7, 1922.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

ANDERSEN (A. F.). Studier over Cesàro's summabilitetsmetode. Med særlig anvendelse overfor potensrækkernes teori. København, Gjellerup, 1921. 8vo. 100 pp.

ANTOINE (L.). 1: Sur l'homéomorphie de deux figures et de leurs voisinages. 2: Propositions données par la faculté. Thèses présentés à la faculté des sciences de Strasbourg. Paris, Gauthier-Villars, 1921. 4to. 105 pp.

ARCHIMEDES. Ueber Spiralen. Uebersetzt und mit Anmerkungen und einem Anhang versehen von A. Czwalina-Allenstein. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1922. 8vo. 71 pp.

BOREL (E.). Méthodes et problèmes de la théorie des fonctions. Paris, Gauthier-Villars, 1922. 8vo. 148 pp.

BRUNSCHVICG (L.). L'expérience humaine et la causalité physique. Paris, Alcan, 1922. 16 + 625 pp.

CESÀRO (G.). Géométrie et crystallographie. Tous les polyèdres symétriques, non-superposables ou superposables à leurs images, peuvent etre déduits d'un petit nombre de formes très simples, en n'employant que la droite comme élément de symétrie. Bruxelles, Hayez, 1922.

COURANT (C.). See HURWITZ (A.).

CULLIS (C. E.). Chapters on algebra. Being the first three chapters of Matrices and determinoids, volume 3. Cambridge, University Press, 1920. 8vo. 9 + 182 pp.

CZWALINA-ALLENSTEIN (A.). See ARCHIMEDES.

DENDY (A.). Problems of modern science. A series of lectures delivered at King's College, University of London. By various authors. Edited by A. Dendy. London, Harrap, 1922. 237 pp.

DE ECHAGUIBEL (E.). Principios de análisis matematico. El problema fundamental de análisis. Bilbao, Eléxpuru Hermanos, 1922. 8vo. 286 pp.

EULER (L.). Opera omnia. Series I: Opera mathematica. VIII: Introductio in analysin infinitorum. Tomus primus. Ediderunt Adolph Krazer et Ferdinand Rudio. Leipzig, Teubner, 1922. 12 + 392 pp.

FREUND (M.). Ueber das Potential mehrfach belegter Flächen. (Diss., Halle-Wittenberg.) Halle, Buchdruckerei Hohmann, 1922. 39 pp.

HAAG (J.). Cours de mathématiques spéciales. Exercices du Tome II (Géometrie analytique). Paris, Gauthier-Villars, 1922. 502 pp.

HESSENBERG (G.). Vom Sinn der Zahlen. Tübingen, Schnürlen, 1922.

HURWITZ (A.). Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Herausgegeben und ergänzt durch einen Abschnitt über Geometrische. Funktionentheorie von C. Courant. Berlin, Springer, 1922. 12 + 400 pp.

KRAZER (A.). See EULER (L.).
RUDIO (F.). See EULER (L.).

II. ELEMENTARY MATHEMATICS

AMODEO (F.). Complementi di analisi algebrica elementare. Parte 2 del volume 2 degli Elementi di matematica. 3a edizione, migliorata ed aumentata. Napoli, Pierro, 1920. 8vo. 404 + 32 pp.

CLAPHAM (C. B.). Metric system for engineers. London, Chapman and Hall, 1921. 12 + 181 pp.

DAKIN (A.). Examples in practical mathematics. London, Bell, 1922. 64 pp.

FRICKE (R.). Lehrbuch der Differential- und Integralrechnung. 2ter Band: Integralrechnung. 3te Auflage. Leipzig, 1921.

HARPER (H. D.). See WENTWORTH (G.).

HEILAND (F.). Sammlung von Aufgaben aus der ebenen und sphärischen Trigonometrie. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 8vo. 152 pp.

KEAL (H. M.) and others. Tables for technical mathematicians. London, Chapman and Hall, 1922. 8vo. 85 pp.

MICHEL (F.) et POTRON (M.). La composition de mathématiques dans l'examen d'admission á l'Ecole Polytechnique de 1901 a 1921. Paris, Gauthier-Villars, 1922. 8vo. 452 pp.

OSGOOD (W. F.). Introduction to the calculus. New York, Macmillan, 1922. 10 + 449 pp.

POTRON (M.). See MICHEL (F.).

RAABE (E.) und SÖHNER (A.). Aktuelles Welt-Rechenbuch "Summablitz." 4te Ausgabe, herausgegeben von A. Söhner und J. Themessl. Berlin-Karlshorst, Dr. A. Söhner, 1920.

RINGUEZ (R.). Tables de coefficients à l'usage des commerçants et des industrielles. Paris et Liège, Béranger, 1922. 8vo. 45 pp.

SALPETER (J.). Einführung in die höhere Mathematik für Naturforscher und Aerzte. 2te, verbesserte und vermehrte Auflage. Jena, 1921.

SILBERSTEIN (L.). Bell's mathematical tables, with a collection of formulae, definitions, and theorems. London, Bell, 1922. 11 + 250 pp.

SMITH (A. S.). Compound interest, as exemplified in the calculation of annuities immediate and deferred, present values and amounts, insurance premiums, repayment of loans, capitalization of rentals and incomes, etc., with annuity tables. London, Effingham Wilson, 1922. 63 pp.

SMITH (D. E.). See WENTWORTH (G.).

SÖHNER (A.). See RAABE (E.).

STUYVAERT (M.). Algèbre. Premier degré. Gand, van Rysselberghe et Rombaut, 1922. 8vo. 117 pp.

THEMESSL (J.). See RAABE (E.).

WENTWORTH (G.), SMITH (D. E.) and HARPER (H. D.). Machine shop mathematics. Boston, Ginn, 1922. 8vo. 5 + 162 pp.

III. APPLIED MATHEMATICS

ALLIEVI (L.). Théorie du coup de bélier. Notes I à V. Traduite par D. Gaden. Préface de R. Neeser. Paris, Dunod, 1921. 16 + 136 pp.

BARNARD (R. A. J.). Elementary statics of two and three dimensions. London, Macmillan, 1921. 7 + 254 pp.

BAUER (E.). La théorie de Bohr, la constitution de l'atome et la classification périodique des éléments. Paris, Hermann, 1922. 8vo. 52 pp.

BECQUEREL (J.). Le principe de relativité et la théorie de la gravitation. Paris, Gauthier-Villars, 1922. 8vo. 342 pp.

BERTHOUD (A.). La constitution des atomes. Paris, Payot, 1922. 16mo. 158 pp.

BOURGEOIS (R.) et NOIREL (H.). Géodésie élémentaire. 2e édition. Paris, Doin, 1922. 16mo. 460 pp.

BOUTHILLON (L.). La théorie et la pratique des radiocommunications. II: La propagation des ondes électromagnétiques à la surface de la terre. Paris, Delagrave, 1921. 8vo. 15 + 340 pp.

CAMPBELL (N. R.). Physics. The elements. Cambridge, University Press, 1920. 10 + 566 pp.

CHALLÉAT (J.) et THOMAS (A.). Mécanique des affûts. 2e édition. Tome 1. Paris, Doin, 1922. 16mo. 360 pp.

CHRISTESCO (S.). Explorations dans l'ultra-éther de l'univers et les anomalies des théories d'Einstein. Paris, Alcan, 1922. 8vo. 440 pp.

COLLIER (W. A.). Einführung in die Variationsstatistik mit besonderer Berücksichtigung der Biologie. Berlin, 1921.

DAVIDGE (H. T.) and HUTCHINSON (R. W.). Technical electricity. 4th edition. London, University Tutorial Press, 1922. 12 + 514 pp.

DEVILLERS (R.). Dynamics of the aeroplane. Translated by W. J. Walker. London, Spon, 1922. 302 pp.

DORNO (C.). Physik der Sonnen- und Himmelsstrahlung. Braunschweig, Vieweg, 1919. 8 + 126 pp.

EHRINGHAUS (A.). Das Mikroscop. Seine wissenschaftlichen Grundlagen und seine Anwendungen. (Aus Natur und Geisteswelt, Nr. 678.) Leipzig, 1921.

ELLIS (C. A.). Essentials in the theory of framed structures. New York, McGraw-Hill, 1922. 15 + 330 pp.

FAJANS (K.). Radioaktivität und die neueste Entwicklung der Lehre von den chemischen Elementen. 4te erweiterte Auflage. Braunschweig, Vieweg, 1922. 137 pp.

GADEN (D.). See ALLIEVI (L.).

GALETTI DI CADILHAC (R. C.). The framework of wireless telegraphy. A new theory of electric wave sources and propagations, with rational applications. London, Crosby Lockwood and Son, 1921. 48 pp.

GALLIOT (F.). See JACQUINOT (O.).

GLAZEBROOK (R.). A dictionary of applied physics. Volume 2: Electricity. London, Macmillan, 1922. 8vo. 7 + 1104 pp.

GOSSOT (—.) et LIOUVILLE (M.). Balistique intérieure. Paris, Baillière, 1922. 8vo. 446 pp.

GROSSMANN (M.). Darstellende Geometrie. 2te, umgearbeitete Auflage. 2ter Teil. Leipzig, Teubner, 1921.

GUTHNICK (B.). See SCHEINER (J.).

HALL (G. L.). Elementary theory of alternate current working. 2d edition. London, Benn, 1921. 196 pp.

HAWKINS (C. C.). The dynamo. Its theory, design and manufacture. 6th edition, revised throughout and largely rewritten. Volume 1. London, Pitman, 1922. 24 + 615 pp.

HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. Berlin, 1922.

HOCHMUTH (K.). Der Kreiselkompass. Leipzig, 1921.

HOUSSAY (F.). Force et cause. Paris, Flammarion, 1920. 247 pp.

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ISENKRAHE (C.). Zur Elementaranalyse der Relativitätstheorie. Einleitung und Vorstufen. Braunschweig, Vieweg, 1921. 5 + 133 pp.

JACQUINOT (O.) et GALLIOT (F.). Navigation intérieure. Canaux. Paris, Gauthier-Villars, 1922. 8vo. 600 pp.

KÖHLER (W.). Die physischen Gestalten in Ruhe und in stationären Zustand. Braunschweig, Vieweg, 1920. 20 + 263 pp.

LANDSBERG (T.). Das Verfahren der Einflusslinien. 7te, verbesserte Auflage. Berlin, 1920.

LEA (F. C.). Elementary hydraulics for technical students. London, Arnold, 1922. 8 + 224 pp.

LEHMANN (W.). Energie und Entropie. Berlin, Springer, 1921. 40 pp.

LESLIE (G. H.). See THOMPSON (G.).

LIOUVILLE (M.). See GOSSOT (—.).

LORENZ (R.). Raumerfüllung und Ionenbeweglichkeit. Leipzig, Voss, 1922. 6 + 289 pp.

MACH (E.). Die Prinzipien der Wärmelehre. Historisch-kritisch entwickelt. 3te Auflage. Leipzig, Barth, 1919. 12 + 484 pp.

MAGNEL (G.). Calcul pratique des poutres continues de béton armé en tenant compte de la raideur des colonnes. Gand, 1922. 58 pp.

MAUDUIT (A.). Machines électriques. Théorie, essais et construction. 2e édition. Paris, Dunod, 1922. 28 + 1180 pp.

MARCHAND BEY (E. E.). Nouveaux principes d'hydraulique générale et de mécanique d'après une étude nouvelle de la contraction. Livry-sur-Seine, chez l'auteur, et Paris, Desforges, 1921. 8vo. 96 pp.

MAYER (R.). Die Knickfestigkeit. Berlin, Springer, 1921. 8vo. 500 pp.

MELDRUM (A. N.). The development of the atomic theory. London, Oxford University Press, 1921. 2 + 13 pp.

MIE (G.). La théorie Einsteinienne de la gravitation. Essai de vulgarization. Ouvrage traduit de l'allemand. Paris, Hermann, 1922. 12 +119 pp.

MOREUX (T.). Origine et formation des mondes. Paris, Doin, 1922. 12 + 401 pp.

MÖRL (A.). Das Wesen der Strahlung. Innsbruck, Wagner, 1921. 24 pp.

MURNAGHAN (F. D.). Vector analysis and the theory of relativity. Baltimore, Johns Hopkins Press, 1922. 10 + 125 pp.

NEESER (R.). See ALLIEVI (L.).

NEUMANN (H.). See PETER (B.).

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NOIREL (H.). See BOURGEOIS (R.).

NYS (D.). La notion d'espace. Bruxelles, Editions Robert Sand, 1922. 8vo. 448 pp.

PAPELIER (G.). Précis de géométrie descriptive. Paris, Vuibert, 1919. 8vo. 316 pp.

PETER (B.). Die Planeten. 2te Auflage, durchgesehen von H. Neumann. Leipzig, Teubner, 1920.

PLANCK (M.). Vorlesungen über Thermodynamik. 6te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 10 + 292 pp.

PRÖLL (A.). Flugtechnik. Grundlagen des Kunstfluges. München, 1919.

RICHARDSON (L. F.). Weather predictions by numerical process. Cambridge, University Press, 1922. 12 + 236 pp.

SAUGER (M.). Einstein ou Euclide. Paris, Imprimerie des Presses universitaires de France, 1922. 12mo. 21 pp.

SCHEINER (J.). Der Bau des Weltalls. 5te Auflage. Bearbeitet von B. Guthnick. Leipzig, Teubner, 1920.

SOPER (H. E.). Frequency arrays, illustrating the use of logical symbols in the study of statistical and other distributions. Cambridge, University Press, 1922. 8vo. 48 pp.

STATON (W.). Magnetism and electricity. London, Cassell, 1922. 8vo. 160 pp.

STEVENSON (E. L.). Terrestrial and celestial globes, their history and construction, including a consideration of their value as aids in the study of geography and astronomy. New Haven, Yale University Press, published for the Hispanic Society of America, 1921. 2 volumes. 26 + 218 + 12 + 292 pp.

THOMAS (A.). See CHALLÉAT (J.).

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WALTE (W.). Einstein, Michelson, Newton. Die Relativitätstheorie, Wahrheit und Irrtum. Hamburg, W. Gente, 1921. 47 pp.

WALKER (W. J.). See DEVILLERS (R.).

WASHBURN (E. W.). An introduction to the principles of physical chemistry from the standpoint of modern atomistics and thermodynamics. New York, McGraw-Hill, 1921. 18 + 518 pp.

ZORETTI (L.). Cours de cinématique appliquée, professé aux élèves de première année. Caen, Institut technique de Normandie, 1921. 8vo. 64 pp.

THIRTY-FIRST ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

ALEXANDER, J. W. On transformations with invariant points. Read Sept. 8, 1921. *Transactions of this Society*, vol. 23, No. 1, pp. 89–95; Jan., 1922.

BEATTY, S. The algebraic theory of algebraic functions of one variable. Read Dec. 28, 1921. *Proceedings of the London Mathematical Society*, (2), vol. 20, No. 6, pp. 435–449; Feb., 1922.

BELL, E. T. Quelques conséquences arithmétiques de l'équation à trois termes dans la théorie des fonctions elliptiques. Read (San Francisco) June 18, 1920. *Giornale di Matematiche*, vol. 59, pp. 93–114; Jan.–June, 1921.

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